The Multispectral Turbulence Decimation Method

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Outline

- High Reynolds-Number Turbulence
- Shell Models of Turbulence
- Spectral Reduction
- Multispectral Reduction
  - Arrangement of Decimated Grids
  - Projection and Prolongation Between Grids
- Multispectral Simulations
- Conclusions

High Reynolds Number Turbulence

- Turbulent systems are characterized by the Reynolds number

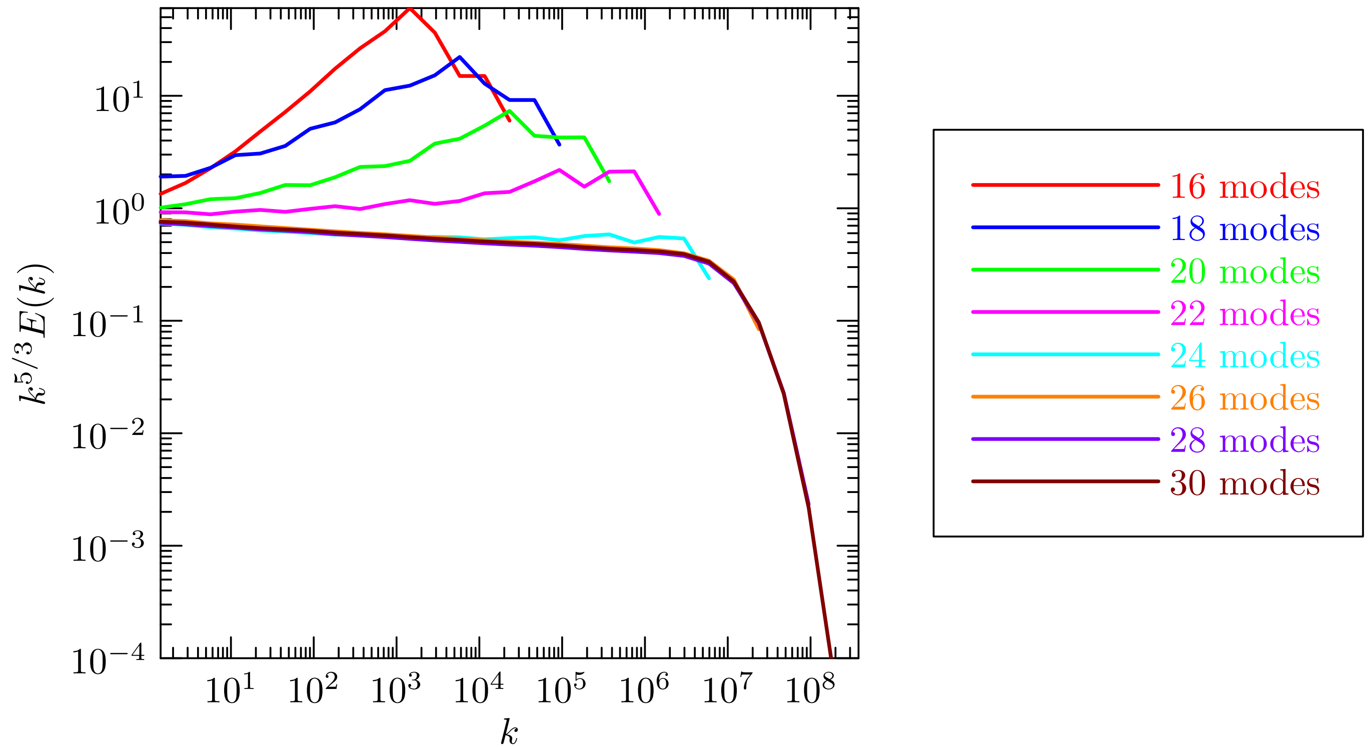
$$R = \frac{UL}{\nu},$$

  where U and L are a characteristic velocity and length scale, and ν is the (kinematic) viscosity.

- The required resolution grows as $R^{9/4}$.
- Airplanes have $R \approx 10^6 \Rightarrow 10^{16}$ modes.
- The atmosphere has $R \approx 10^9 \Rightarrow 10^{20}$ modes.
- Jupiter's Red Spot has $R \approx 10^{14} \Rightarrow 10^{31}$ modes.
- The state of the art is $4096^3 \approx 10^{10}$ modes.

High Reynolds Number Turbulence

- Under resolved simulations can have errors even at large scales:

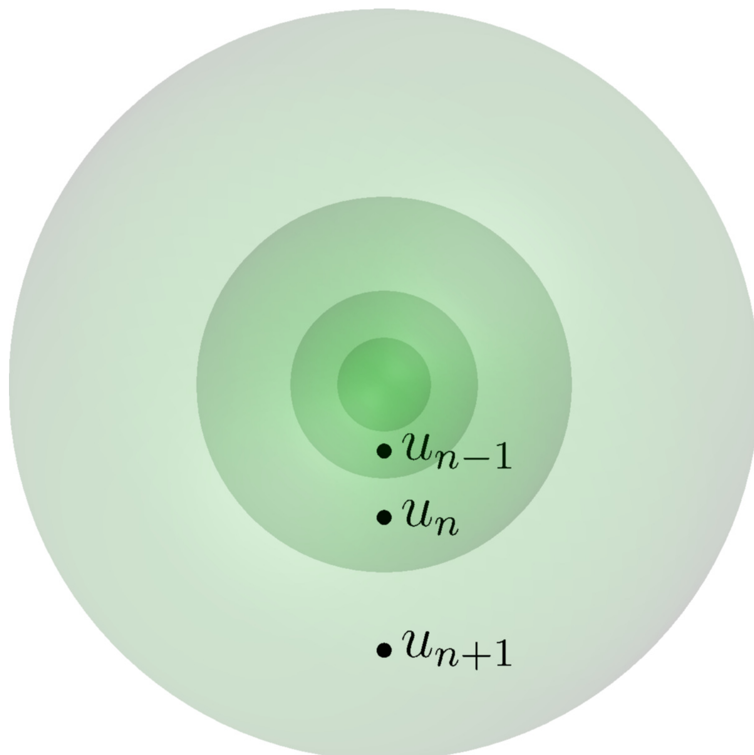


- Since we are unable to perform full-resolution simulations, we have two choices:
- increase the viscosity;
- use a subgrid model to approximate the effect of the small scales.

Shell Models of Turbulence

- Shell models are systems of ODEs which mimic the Fourier-transformed Navier–Stokes equation.

- Collections of modes $\{\boldsymbol{u}_{\boldsymbol{k}} : k \in [\lambda^n, \lambda^{n+1})\}$ are represented by a single quantity u_n:



Shell Models of Turbulence: Interaction

- The convolution is replaced with a quadratic function of u:

$$\frac{du_n}{dt} = k_n \sum_{p,q} c_{p,q} u_p u_q - \nu k_n^2 u_n.$$

- The DN model [Desnyansky & Novikov 1974] has nearest-neighbour interactions and conserves energy $E \doteq \frac{1}{2}\sum |u_n|^2$:

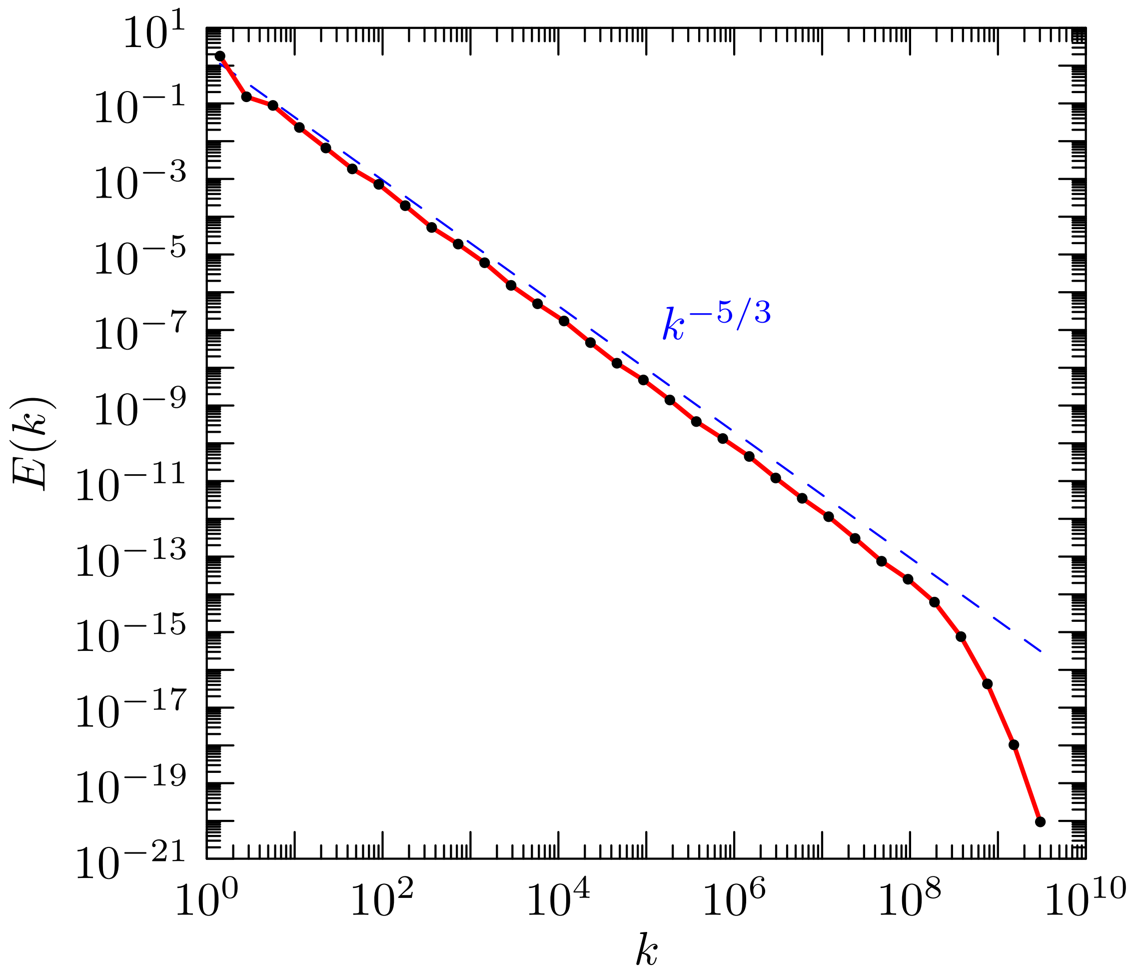
$$\frac{du_n}{dt} = ik_n\big(a_n u_{n-1}^2 - \lambda a_{n+1} u_n u_{n+1} + b_n u_{n-1} u_n - \lambda b_{n+1} u_{n+1}^2\big)^* - \nu k_n^2 u_n.$$

- The GOY [Gledzer 1973], [Yamada & Ohkitani 1987] model adds next-nearest-neighbour interactions and also conserves the helicity-like quantity $H = \frac{1}{2}\sum_n (-1)^n k_n |u_n|^2$:

$$\frac{du_n}{dt} = ik_n\left(\alpha u_{n+1} u_{n+2} + \frac{\beta}{\lambda} u_{n-1} u_{n+1} + \frac{\gamma}{\lambda^2} u_{n-1} u_{n-2}\right)^* - \nu k_n^2 u_n.$$

Shell Models: Kolmogorov Scaling

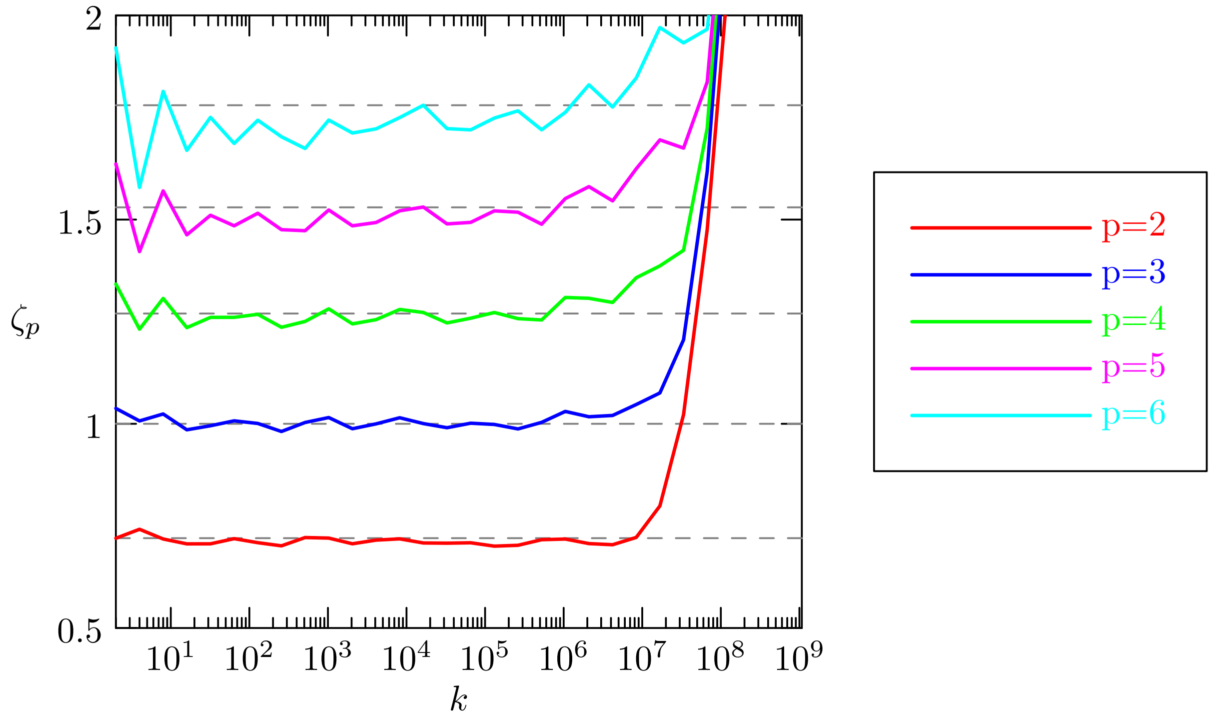
- Simulations reproduce a $k^{-5/3}$ Kolmogorov inertial range:



- Shell models are simpler and easier to simulate than the Navier–Stokes equations [Bowman *et al.* 2006].

Shell Models: Intermittency

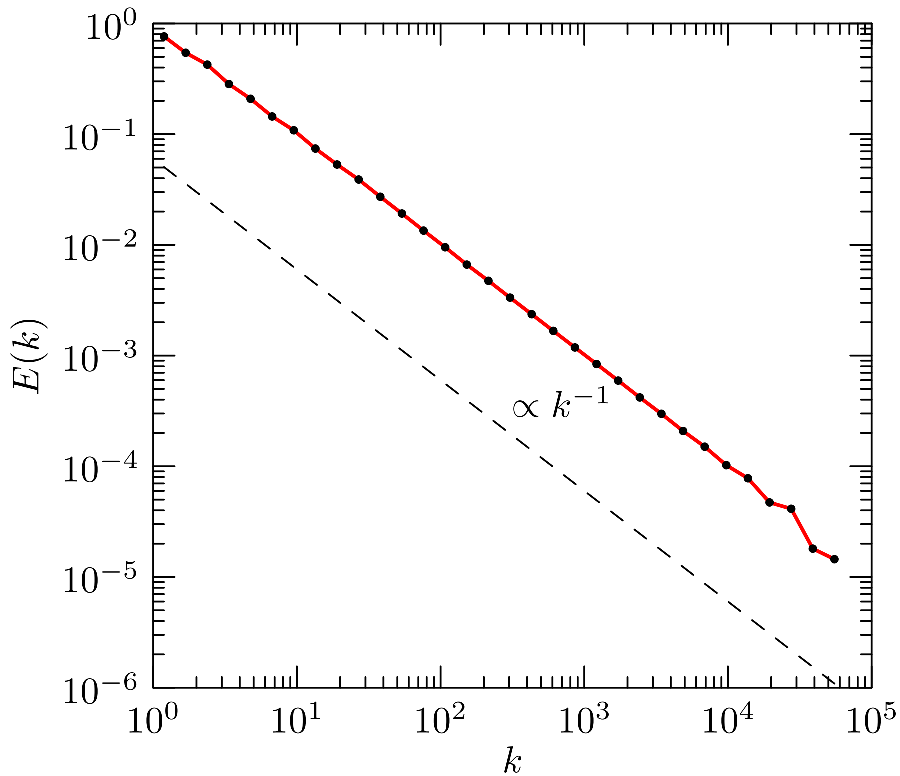
- They also reproduce statistical properties of Navier–Stokes turbulence: the moments $\langle |u_n|^p \rangle \sim k_n^{-\zeta_p}$



scale very much like experimental structure exponents for 3D turbulence (dashed lines) [Herweijer & van de Water 1995].

Shell Models: Equipartition

- Another property shell models share with the Navier–Stokes equation is the equipartition spectrum.
- With no force term and setting $\nu = 0$, the system reaches equipartition, with $E(k) \sim 1/k$.



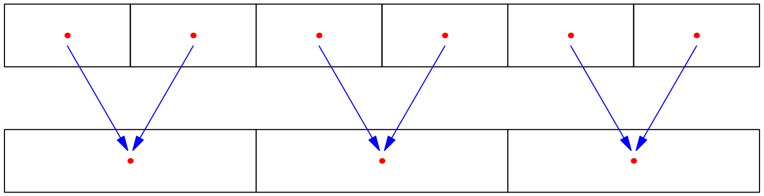
- This provides a useful test of the numerical implementation.

Spectral Reduction

- Navier–Stokes simulations at high Reynolds number require more modes than current computers can handle.
- We use shell models as testbeds for developing numerical techniques.
- Consider a generalization of spectral reduction [Bowman *et al.* 1999]: Instead of evolving u_n directly, we evolve

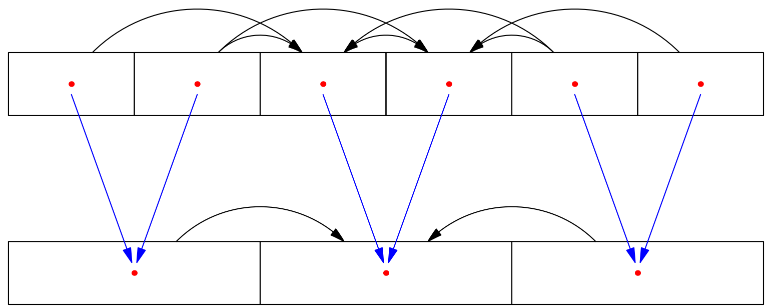
$$u_{n,1} \doteq \frac{u_{2n} + \sigma_n^* u_{2n+1}}{1 + |\sigma_n|^2}, \qquad \sigma_n \doteq \frac{u_{2n+1}}{u_{2n}}.$$

- Then $u_{2n} = u_{n,1}$ and $u_{2n+1} = \sigma_n u_{n,1}$.
- This reduces the number of active modes by half:



Fixed Point

- Spectral reduction reduces the GOY model to the DN model, which is a fixed point.



- Further reduction is straightforward:

$$u_{n,\ell+1} \doteq \frac{u_{2n,\ell} + \sigma^*_{n,\ell} u_{2n+1,\ell}}{1 + |\sigma_{n,\ell}|^2}, \quad \sigma_{n,\ell} \doteq \frac{u_{2n+1,\ell}}{u_{2n,\ell}}.$$

$$\text{GOY} \xrightarrow{\text{reduction}} \text{DN} \xrightarrow{\text{reduction}} \text{DN} \xrightarrow{\text{reduction}} \text{DN}$$

Decimation

- Spectral reduction provides us with evolution equations for the velocity amplitudes $u_{n,1}$.

- In order to close the equations, we must approximate σ_n.

$$\sigma_n = 1 \quad \Rightarrow \quad u_{n,1} = \frac{u_{2n} + u_{2n+1}}{2},$$

  i.e. we set the decimated mode to be the average of the undecimated modes.

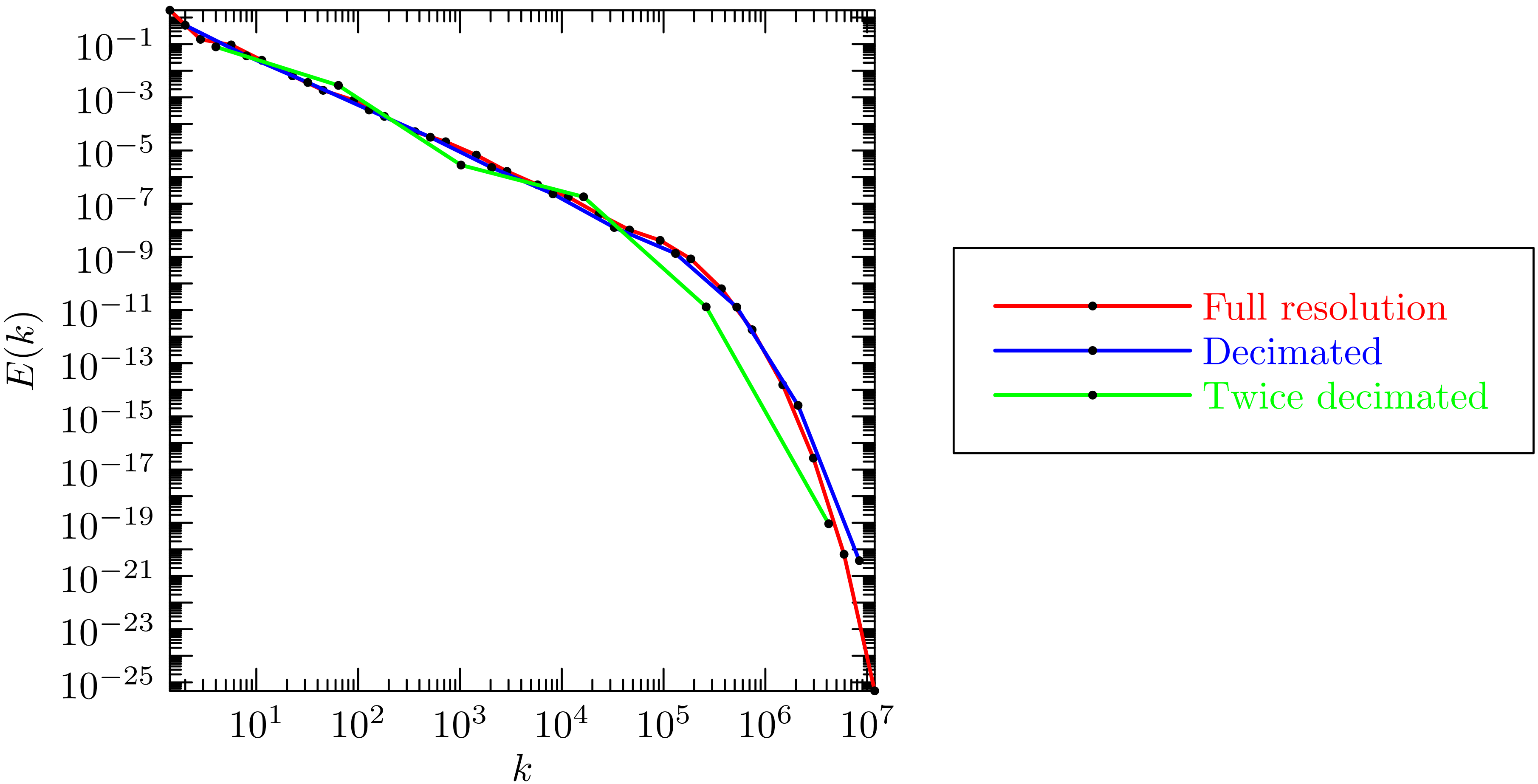
- The energy $E_1 \doteq \frac{1}{2} \sum_n 2 \left|u_{n,1}\right|^2$ is conserved.

- Binning modifies the viscous term and the interaction coefficients:

$$(\alpha, \beta, \gamma) \to (a, b) \doteq \left(\frac{\gamma}{\lambda^2}, -\frac{\alpha}{\lambda}\right) \to \frac{(a, b)}{2}.$$

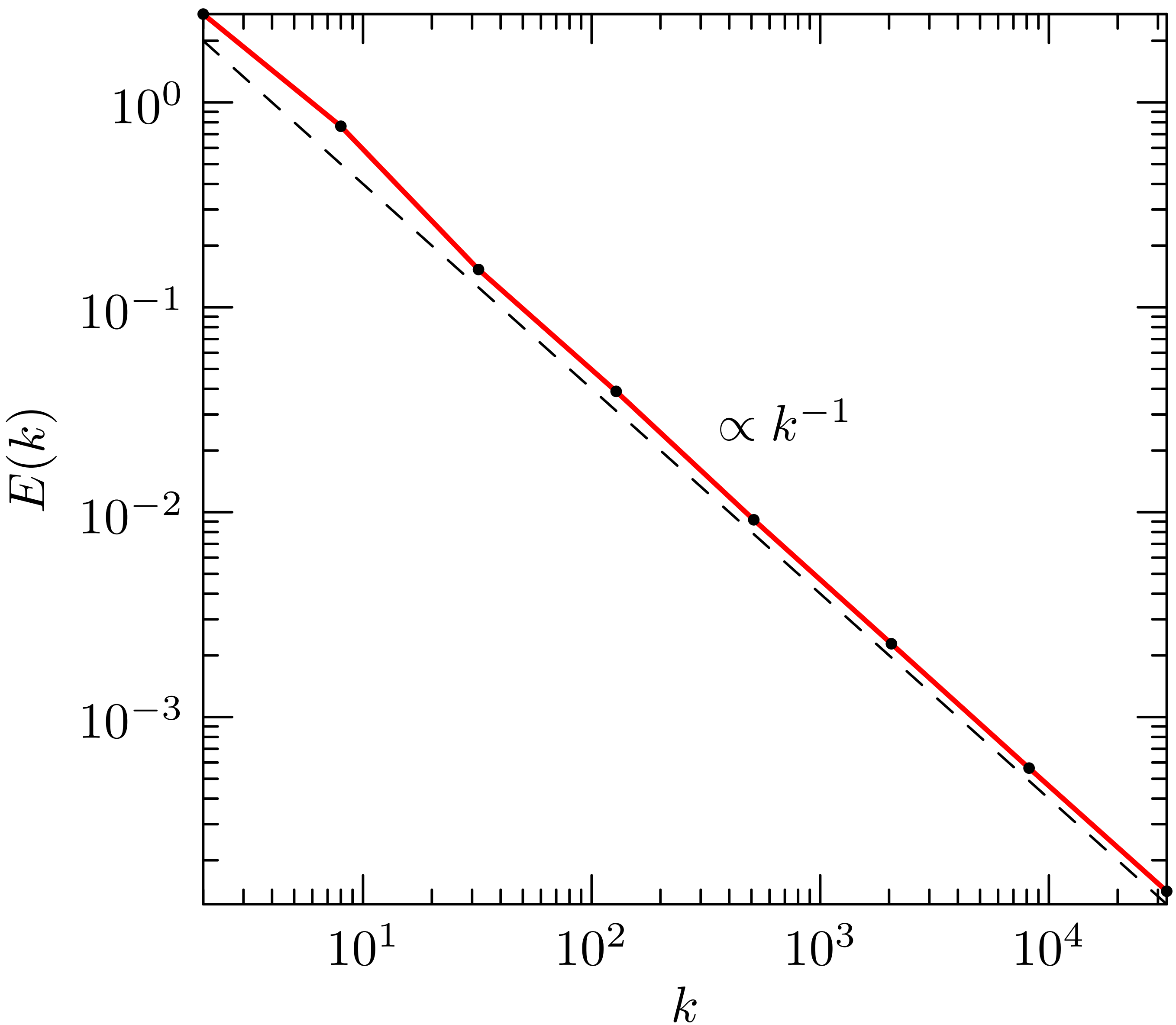
Decimation

- Spectrally reduced shell models reproduce the Kolmogorov spectrum:



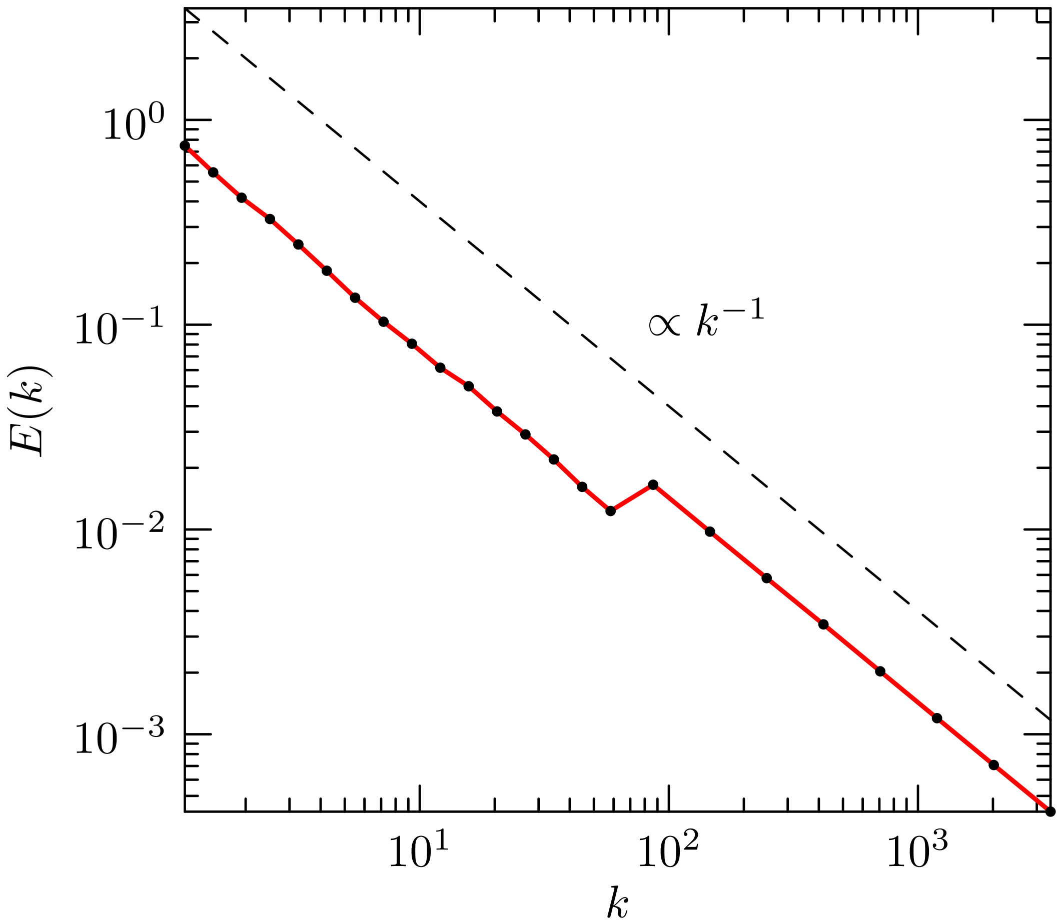
Decimation

- and exhibit equipartition when the bins are equally-spaced:



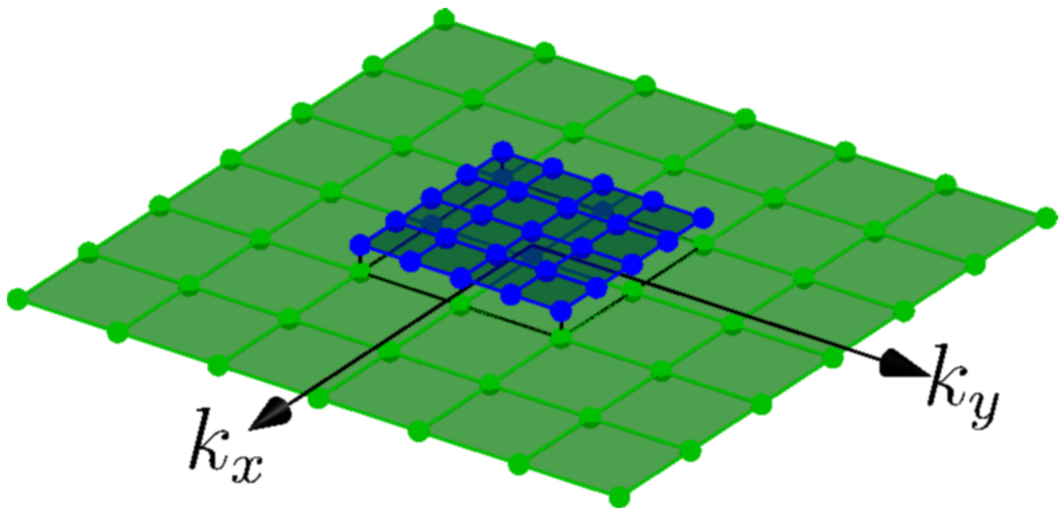
Non-Uniform Spectral Reduction

- Low-wavenumber modes are more physically important.
- Ideally, we would like to decimate only modes with $k > k_{\text{cutoff}}$.
- Unfortunately, this modifies the equipartition spectrum:



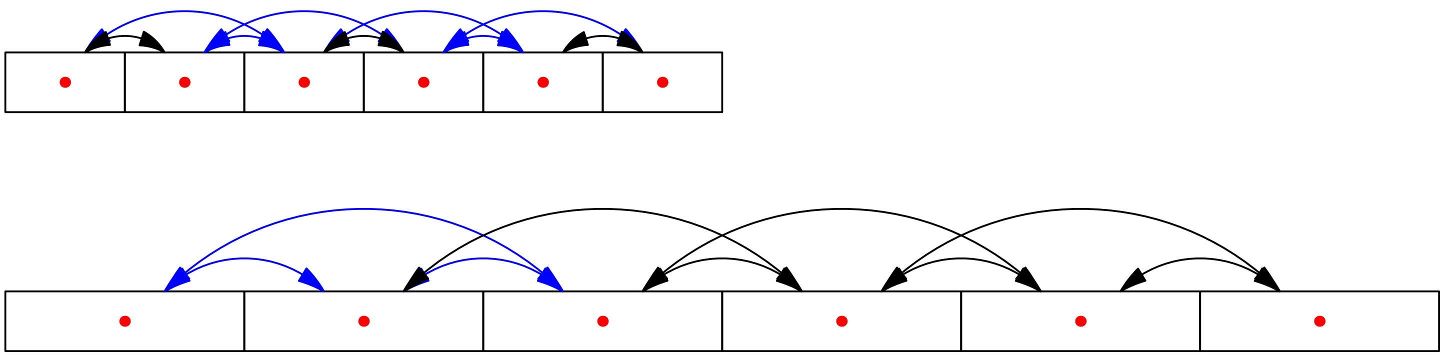
Multispectral Reduction

- The multispectral method is designed to solve these problems.
- We wish to decimate only at high-wavenumbers.
- The equipartition spectrum is only correct on uniform grids.
- Therefore, we must use multiple, differently decimated, partly overlapping grids.

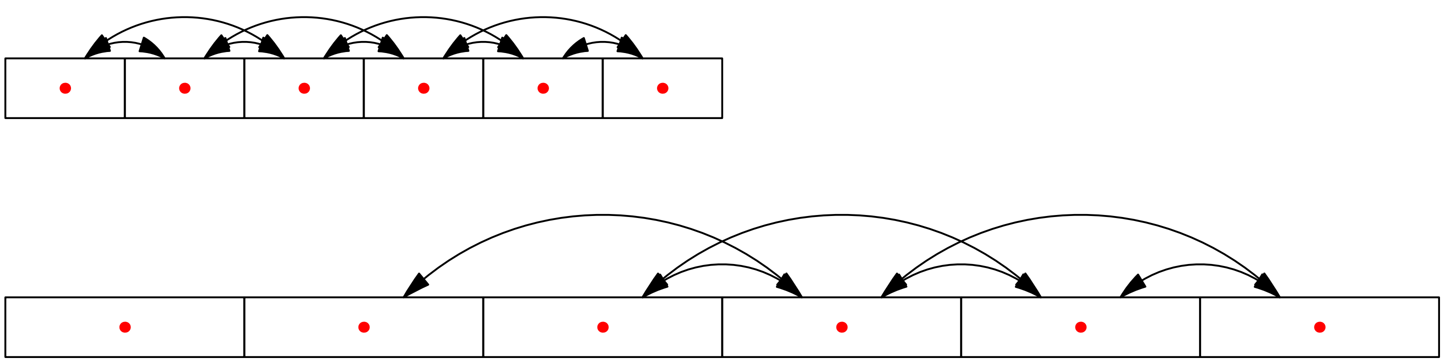


Multispectral Reduction

- All but one grid is decimated using spectral reduction.
- For a general shell model, some of the interactions may be counted twice:

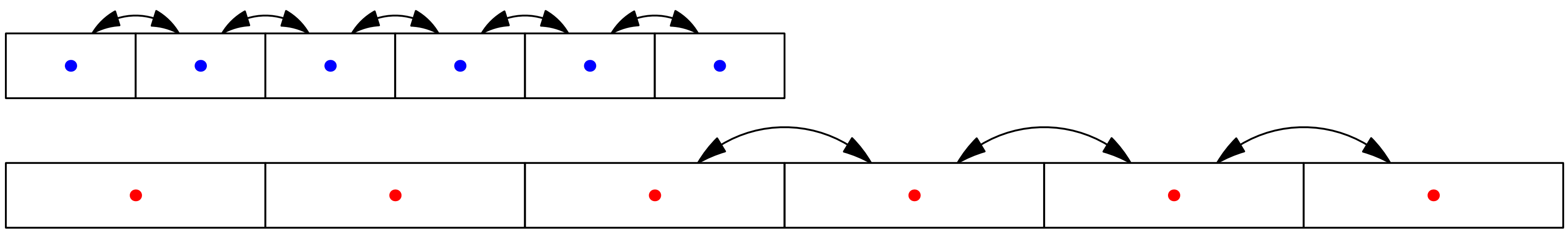


- We remove interactions from the coarse grid to eliminate redundancy.



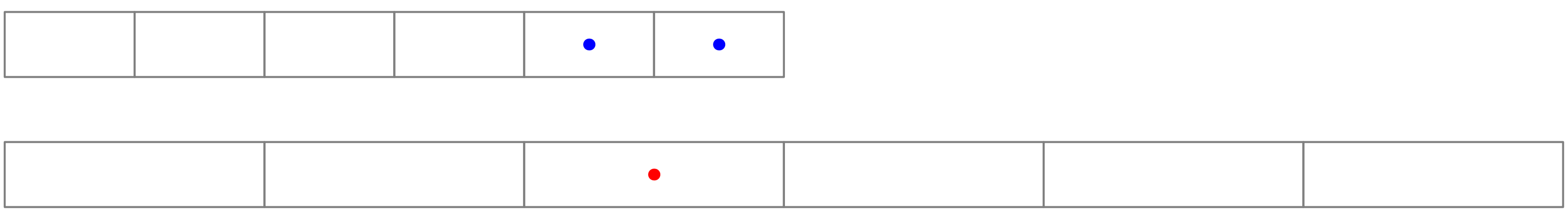
Multispectral Reduction: Grid Geometry

- Example: The DN model has nearest-neighbour interactions:



- The energy of the new system is $\frac{1}{2}\sum_n |u_n|^2\Delta_n$, where we sum over only visible modes and $\Delta_n = 1(2)$ on the fine (coarse) grid.

- There is a triplet of overlapping active modes:



- Projection/prolongation takes the energy input from each mode of the overlapping triplet and scales the modes so that the energies in the high-resolution and low-resolution grids agree.

Multispectral Method: Projection

- The solution is advanced in time as follows:
- At the start of each time step j, the energies of the overlapping modes agree:

$$\frac{1}{2}\left|u_n^j\right|^2 + \frac{1}{2}|u_{n+1}^j|^2 = \frac{1}{2}|u_n^{(1)j}|^2\Delta_n.$$

- Using a Runge–Kutta integrator, the fine grid is advanced in time:

$$u_n^j \to \widetilde{u}_n^{j+1} \quad u_{n+1}^j \to \widetilde{u}_{n+1}^{j+1}.$$

- Next we *project* onto the coarse grid:

$$\widetilde{u}_n^{(1)j} = \sqrt{\frac{\left|\widetilde{u}_n^{j+1}\right|^2 + \left|\widetilde{u}_{n+1}^{j+1}\right|^2}{2}}.$$

Multispectral Method: Prolongation

- Now we advance the coarse grid in time:

$$\widetilde{u}_n^{(1)j} \to u_n^{(1)j+1}.$$

- Finally, we *prolong* from the coarse grid onto the fine grid:

$$u_n^{j+1} = \sqrt{\frac{|u_n^{(1)j+1}|^2}{|\widetilde{u}_n^{(1)j}|^2}}\, \widetilde{u}_n^{j+1}.$$

$$u_{n+1}^{j+1} = \sqrt{\frac{|u_n^{(1)j+1}|^2}{|\widetilde{u}_n^{(1)j}|^2}}\, \widetilde{u}_{n+1}^{j+1}.$$

- The projection and prolongation operators conserve energy whenever the two grids conserve energy in isolation.
- We can also include the changes in phase into the projection and prolongation operators, which may be important for Navier–Stokes turbulence.

Multispectral Method: Normalisation

- In addition to energy conservation, the grids must relax at the same rate.
- We coarse-grain the equations by setting

$$u_n^{(1)} = \frac{u_{2n} + u_{2n+1}}{C}.$$

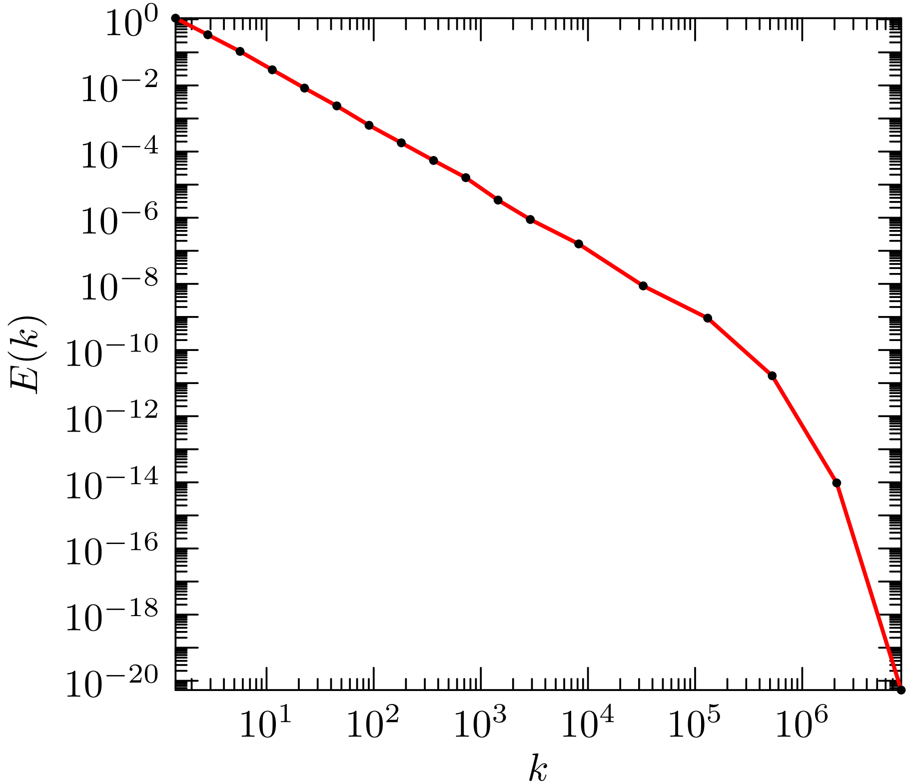
- The phases of u_{2n} and u_{2n+1} are uncorrelated, so

$$\left\langle \left| u_n^{(1)} \right|^2 \right\rangle = \frac{\left\langle \left| u_{2n} + u_{2n+1} \right|^2 \right\rangle}{C^2} = \frac{\left\langle \left| u_{2n} \right|^2 \right\rangle + \left\langle \left| u_{2n+1} \right|^2 \right\rangle}{C^2}$$

$$= \frac{\left\langle \left| u_{2n} \right|^2 \right\rangle + \left\langle \left| u_{2n+1} \right|^2 \right\rangle}{2} \Rightarrow C = \sqrt{2}$$

Multispectral Method: Forced-Dissipative Turbulence

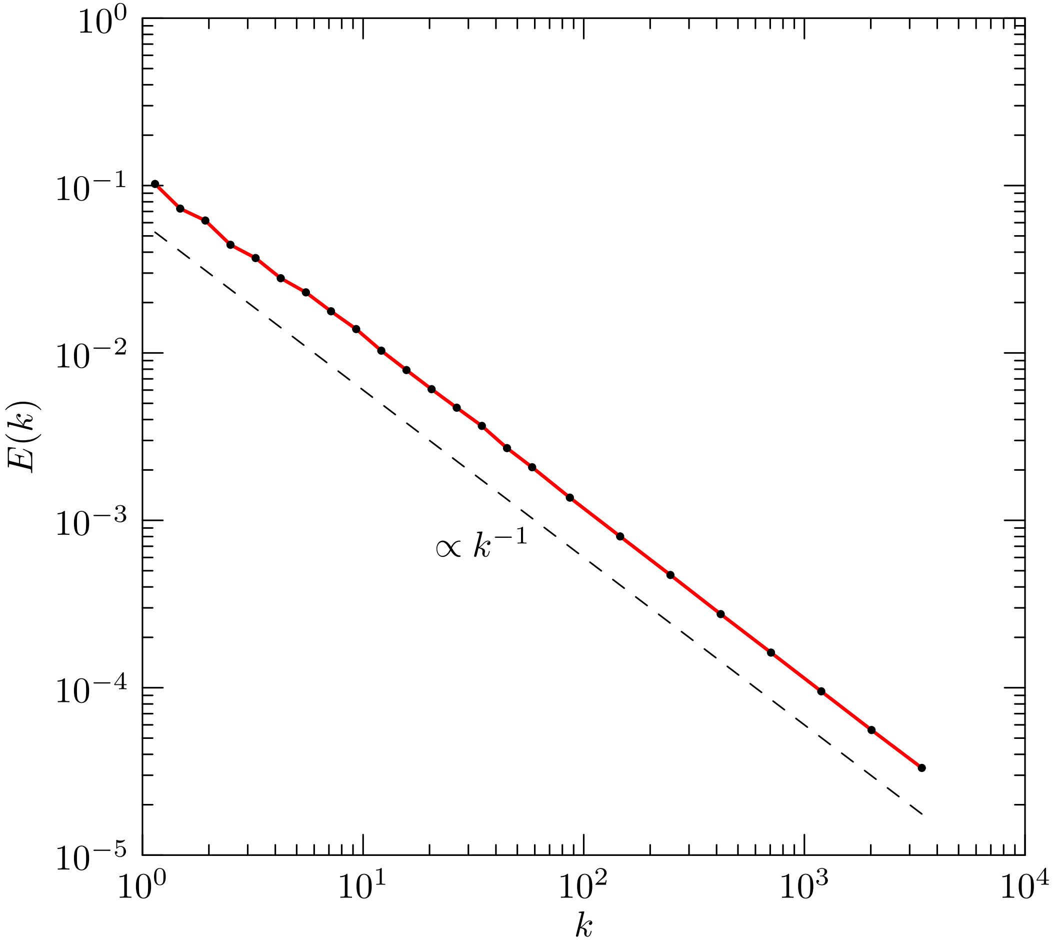
- Multispectral shell models reproduce the Kolmogorov spectrum.



- Note that the resolution change does not disturb the spectrum.

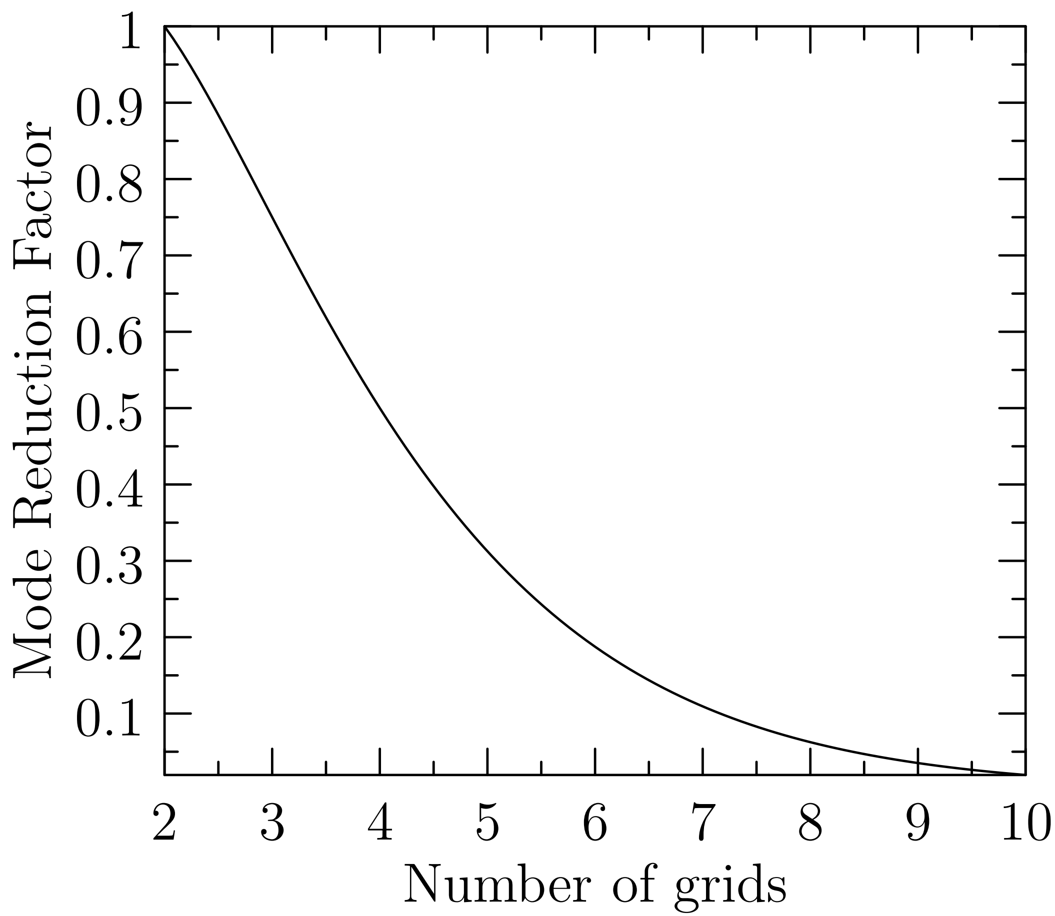
Multispectral Method: Equipartition

- Uniform grids allow us to reach the correct equipartition:



Multispectral Method: Efficiency

- This technique can be extended to a hierarchy of n grids.
- The number of modes is reduced by a factor of $n2^{1-n}$.



- The proportion of the grids that overlap can also be varied.

Conclusions

- Shell models are simple systems that can behave like Navier–Stokes turbulence.

- The multispectral method preserves the behaviour of the full-resolution simulation.

- The multispectral method can be extended to a hierarchy of grids.

- The multispectral method uses fewer modes than a full simulation (by a factor of $n2^{n-1}$).

- Future work: extension of reduction method to 2D and 3D Navier–Stokes dynamic subgrid model.

Asymptote: 2D & 3D Vector Graphics Language

Andy Hammerlindl, John C. Bowman, Tom Prince

`http://asymptote.sf.net`

(freely available under the GNU public license)

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