Limits of Shell Models of Turbulence: Finite-Viscosity Corrections to Kolmogorov Scaling

Malcolm Roberts, University of Alberta; Bruno Eckhardt, Philipps-University Marburg; John Bowman, University of Alberta

Shell models of turbulence are one-dimensional models designed to mimic the Navier–Stokes equations. The two main models are the DN model,

$$\frac{du_n}{dt} = ik_n\left[a\left(u_{n-1}^2 - \lambda u_n u_{n+1}\right) + b\left(\lambda u_{n+1}^2 - u_n u_{n-1}\right)\right]^*,$$

and the GOY model,

$$\frac{du_n}{dt} = ik_n\left(\alpha u_{n+1}u_{n+2} + \frac{\beta}{\lambda}u_{n-1}u_{n+1} + \frac{\gamma}{\lambda^2}u_{n-1}u_{n-2}\right)^*,$$

plus a forcing term and a dissipative term with viscous coefficient ν.

Shell models have the basic form of the spectral Navier–Stokes equations, but are

- one-dimensional,
- set in Fourier space, with $k_n = \lambda^n$,
- and local in wave-number space.

Despite these great differences, they are able to recreate statistical properties of physical turbulence with surprising accuracy. Moreover, the wavenumbers are spaced geometrically, allowing us to simulate systems with Reynolds numbers far beyond what would be otherwise feasible.

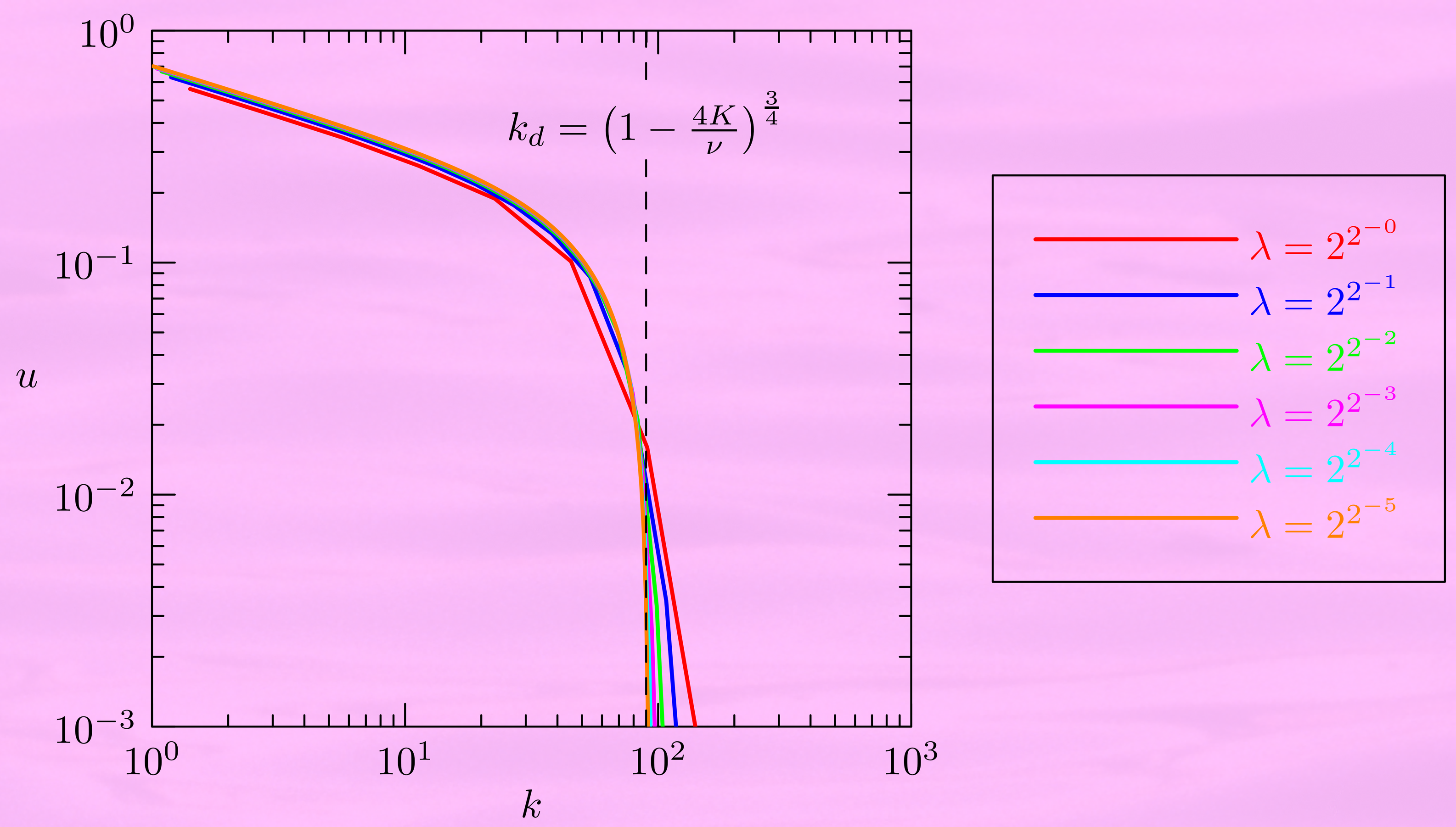
In this poster, we take the limit as the wave-number spacing λ goes to 1, while keeping u constant. Replacing the discrete index n with the continuous variable η, and making the approximation

$$u_{n\pm1} \to u(\eta) \pm \log(\lambda)\frac{\partial u}{\partial \eta} + \mathcal{O}(\log(\lambda)^2) \qquad (1)$$

The nonlinearity scales as $\log\lambda$, which we correct for. This maps both the DN and GOY model to the dynamical system

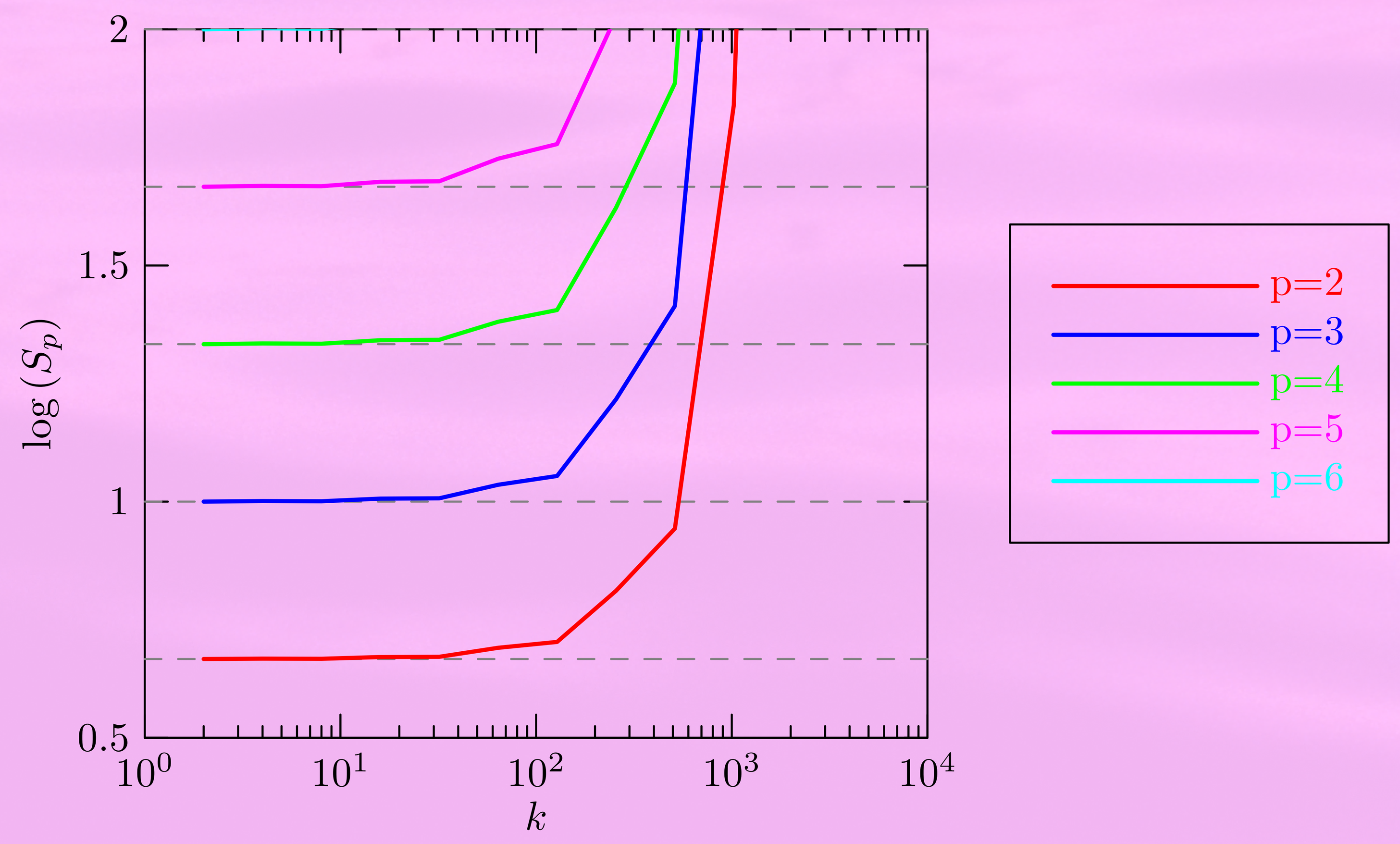
$$\frac{\partial u}{\partial t} + \nu e^{2\eta}u = ie^{\eta}C\left(u^2 + 3u\frac{\partial u}{\partial \eta}\right)^*. \qquad (2)$$

Here, $C = b - a$ for the DN model, and $C = 2\alpha + \beta$ for the GOY model.



Consider the real-valued shell model:

- As $\lambda \to 1$, the computed system approaches the steady-state solution of Equation (2).
- The velocity u reaches zero at the dissipation scale
$$k_d = \left(1 - \frac{4C}{\nu}\right)^{\frac{3}{4}}.$$
- For $\nu \ll 1$, $k_d \sim \nu^{-\frac{3}{4}}$, as per Kolmogorov.



Shell models have moments analogous to higher order statistical moments of Navier–Stokes turbulence:

- Structure functions $S_p \doteq \langle u^p \rangle$.
- Kolmogorov theory predicts: $S_p \sim k^{p/3}$.
- The steady-state solution of Equation (2) yields
$$S_p(k) \sim k^{\frac{p}{3}\left(1+\frac{\nu}{C}\cdot\frac{S_{p-1}(1)}{S_p(1)}\right)},$$
which reduces to Kolmogorov theory when $\nu \ll 1$.

Real-valued turbulence:

- has only viscous corrections to structure functions,
- inviscid, unforced systems do not maximize entropy.

Complex-valued shell turbulence:

- has stronger corrections due to *intermittency*,
- inviscid, unforced systems maximize entropy (are *ergodic*).

These intermittent corrections to the structure functions are not well understood. The correlation between intermittent corrections and ergodicity may indicate a connection.

The finite-viscosity third-order structure function for 3D turbulence is

$$S_3(r) = \frac{4}{5}\epsilon r - 6\nu\frac{d}{dr}\left\langle\left|u(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})\right|^2\right\rangle,$$

where ϵ is the energy infection rate.

The similarity between this and our viscous corrections to shell-model structure functions may indicate that other such corrections exist for 3D turbulence, allowing us to end out poster on a note of optimism which is, as ever, a vital part of turbulence research.