A Multi-Spectral Decimation Scheme for Turbulence Simulations

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Shell Models

- Shell models are reduced models of turbulence formulated in Fourier space.
- The velocities in shell n are replaced by a single quantity, u_n.
- The wavenumbers $k_n = k_0 \lambda^n$ scale geometrically.
- General form:

$$\left(\frac{d}{dt} + \nu k_n^2 \right) u_n = i k_n \sum_{\ell,m} A_{\ell,m} u_\ell^* u_m^* + F_n.$$

- The energy is

$$E = \frac{1}{2} \sum_n |u_n|^2.$$

- If F_n is a white-noise random process, the mean rate of energy injection is $\epsilon = \frac{1}{2} \sum_n \langle |F_n|^2 \rangle$ [Novikov 1964].

DN model

- If we restrict to nearest-neighbour interactions and enforce conservation of energy, the result is a generalised Desnyansky and Novikov [1974] model (DN):

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i k_n \left(a_n u_{n-1}^2 - \lambda a_{n+1} u_n u_{n+1} + b_n u_{n-1} u_n - \lambda b_{n+1} u_{n+1}^2\right)^* .$$

- The nonlinear terms of the DN model have a fixed point:

$$u_n = A k_n^{-1/3}.$$

- For constant coefficients a_n and b_n of opposite sign, Bell & Nelkin [1977] showed that this fixed point is stable.

- This stability is thought to be responsible for the absence of intermittent behaviour in the inviscid DN model.

GOY model

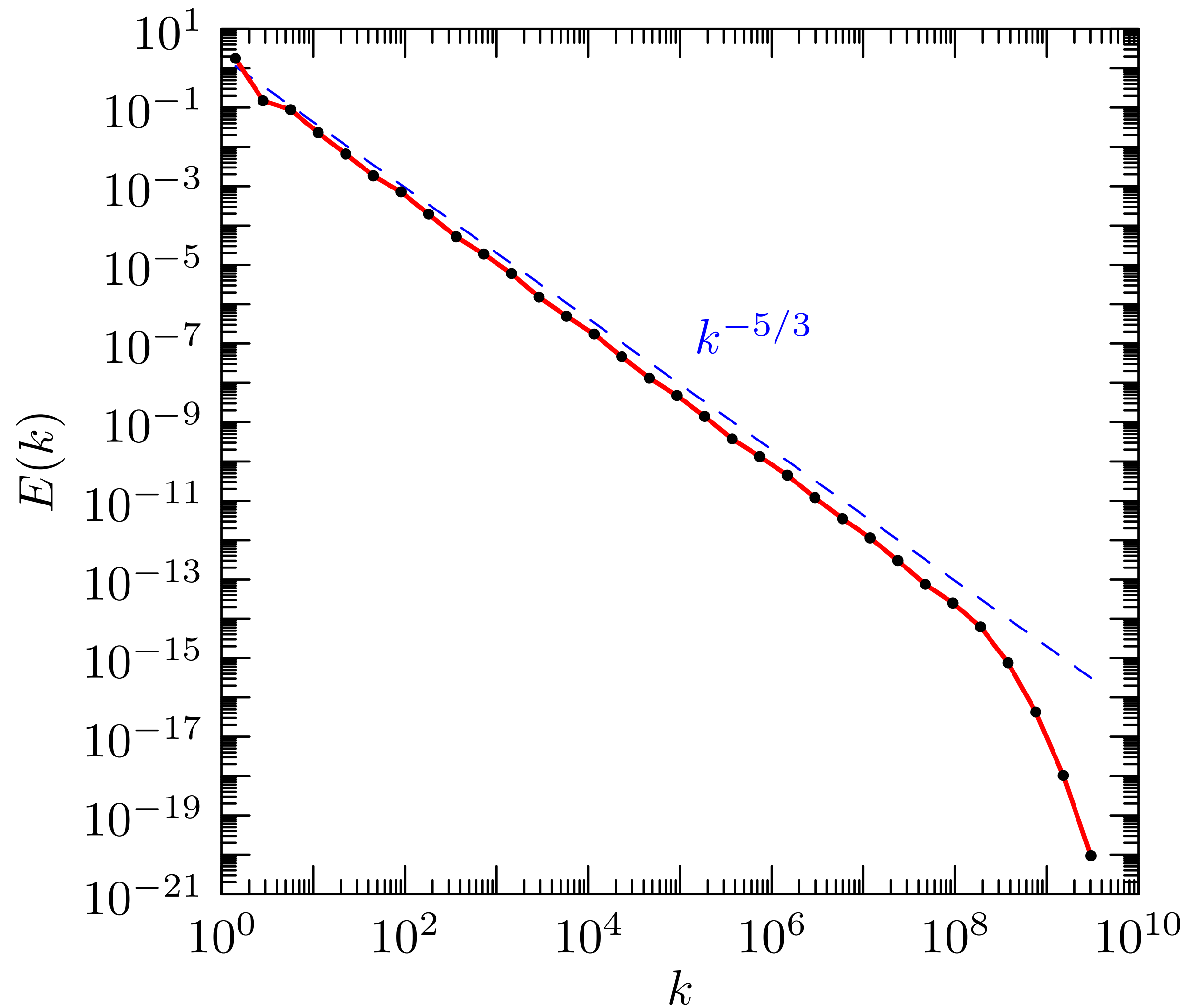
- The GOY model is a shell model of turbulence proposed by Gledzer [1973] (the complex version was proposed by Yamada and Ohkitani [1987]) which exhibits intermittency:

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i k_n \left(\alpha u_{n+1} u_{n+2} + \frac{\beta}{\lambda} u_{n-1} u_{n+1} + \frac{\gamma}{\lambda^2} u_{n-1} u_{n-2}\right)^* + F_n.$$

- The GOY model has next-nearest neighbour interactions and conserves energy if $\alpha + \beta + \gamma = 0$.
- Time can be rescaled so that $\alpha = 1$, leaving 2 free parameters: β and λ.
- The free parameters can be chosen to conserve either:
  - Enstrophy: $\frac{1}{2}\sum_n k_n^2 |u_n|^2 \quad$ (2D),
  - Helicity: $\frac{1}{2}\sum_n (-1)^n k_n |u_n|^2 \quad$ (3D).

Forced-Dissipative GOY turbulence

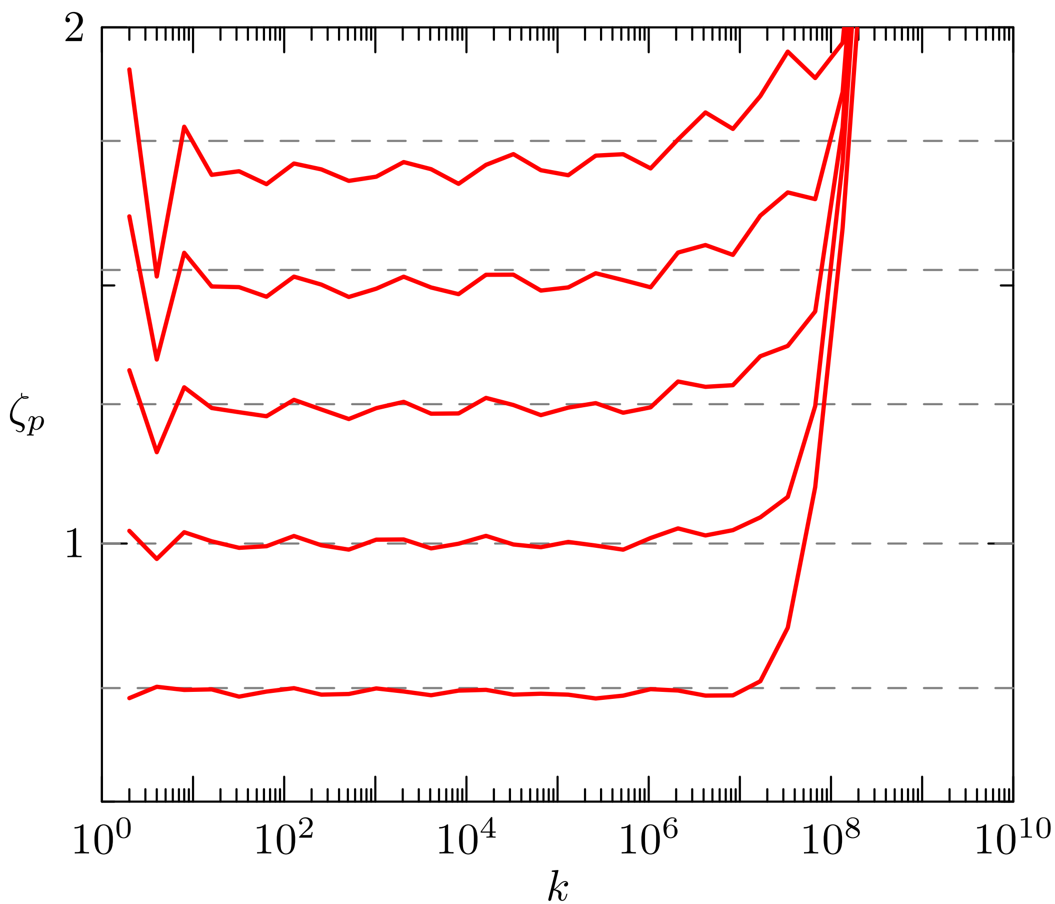
- With forcing on the first shell and small-scale dissipation, we expect a Kolmogorov spectrum:



Structure Functions

$$\langle |u_n|^p \rangle \sim k_n^{-\zeta_p}$$

- For certain choices of parameter, the structure functions for the GOY model demonstrate remarkable agreement with experiment [Herweijer & van de Water 1995].



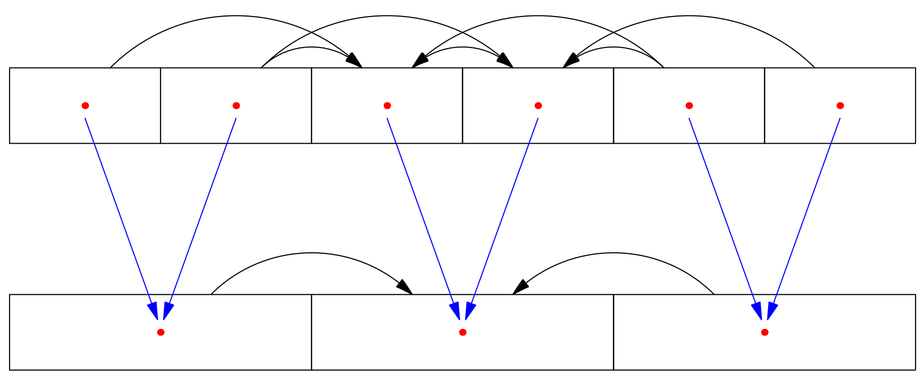
Spectral Reduction of the GOY model

- Apply the method of spectral reduction to the GOY model: set

$$u_n^{(1)} = \frac{u_{2n} + \sigma_n^{(0)*} u_{2n+1}}{1 + |\sigma_n^{(0)}|^2}, \quad \sigma_n^{(0)} = \frac{u_{2n+1}}{u_{2n}},$$

$$\Rightarrow u_{2n} = u_n^{(1)}, \quad \text{and} \quad u_{2n+1} = \sigma_n^{(0)} u_n^{(1)}.$$

- Approximate $\sigma_n^{(0)}$ by the constant $(\langle |u_{n+1}^{(1)}|^2 \rangle / \langle |u_n^{(1)}|^2 \rangle)^{1/4}$.
- This produces nearest-neighbour interactions and nonlinear energy conservation, i.e. the DN model:



Spectral Reduction of the GOY model (cont.)

- This conserves coarse-grained energy,

$$E^{(1)} = \frac{1}{2}\sum_n |u_n^{(1)}|^2 \Delta_n, \quad \Delta_n = (1 + |\sigma_n^{(0)}|^2).$$

- Coefficients for the DN model are given by

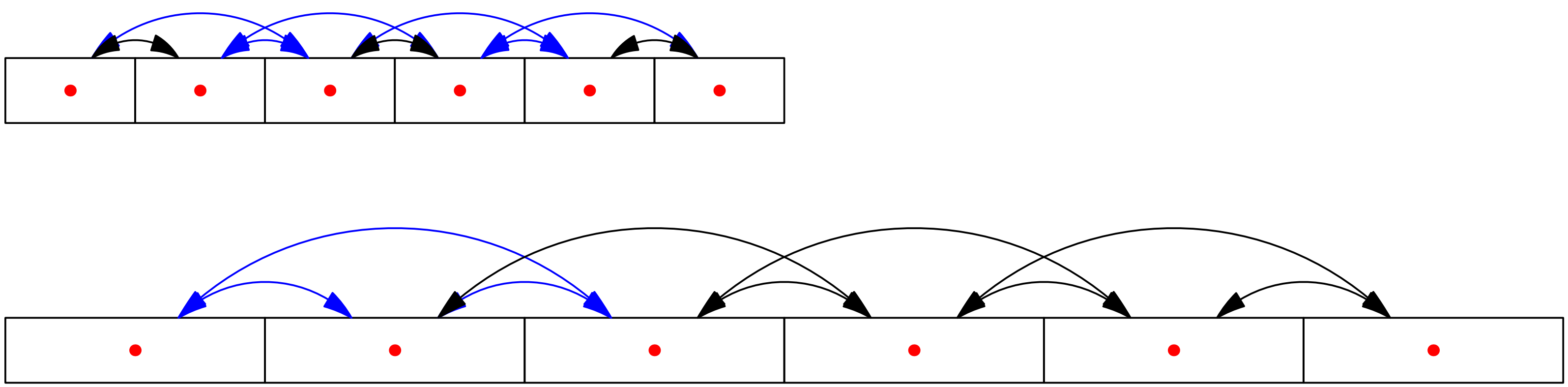
$$\lambda^{(1)} = \lambda^2, \qquad \nu_n^{(1)} = \nu \left(\frac{1 + |\sigma_n^{(0)}|^2 \lambda^2}{1 + |\sigma_n^{(0)}|^2} \right).$$

$$a_n^{(1)} = \frac{\gamma}{\lambda^2} \left(\frac{\sigma_{n-1}^{(0)}}{1 + |\sigma_n^{(0)}|^2} \right), \quad b_n^{(1)} = \frac{-\alpha}{\lambda} \left(\frac{\sigma_{n-1}^{(0)} \sigma_n^{(0)}}{1 + |\sigma_n^{(0)}|^2} \right).$$

- Repeating this approximation on the DN model does not change the form of the governing equation.

Multi-Spectral Reduction

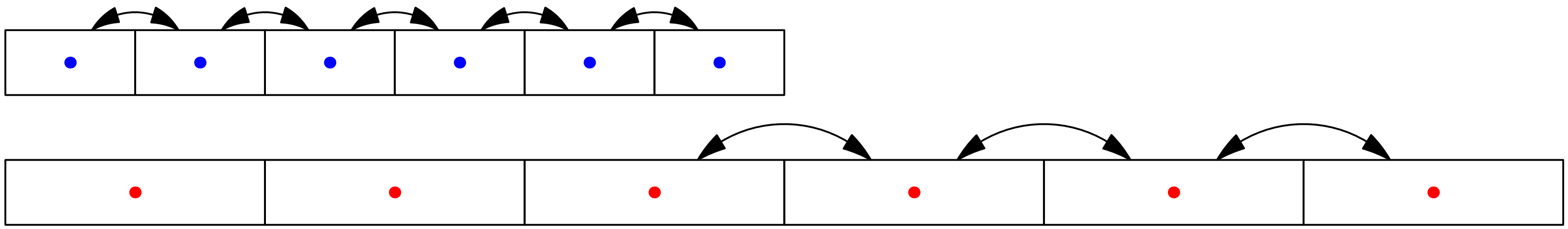
- We use the idea of spectral reduction to do simulations with non-uniform resolution.
- For a general shell model, some of the interactions may be counted twice:



- We remove interactions from the coarse grid to eliminate redundancy.

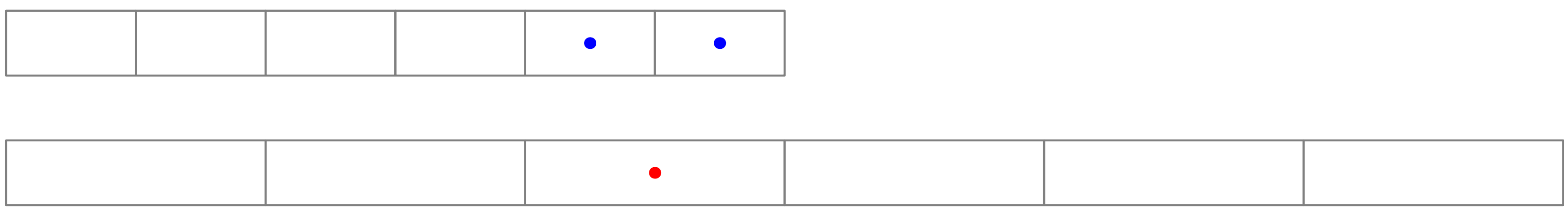
Multi Spectral Reduction: Grid Geometry

- The DN model, which only has nearest-neighbour interactions, leaves a particularly simple picture:



- The energy of the new system is $\frac{1}{2}\sum_n |u_n|^2 \Delta_n$, where we sum over only visible modes and $\Delta_n = 1(2)$ on the fine (coarse) grid.

- There is a triplet of overlapping active modes:



- Projection/prolongation takes the energy input from each mode of the overlapping triplet and scales the modes so that the energies in the high-resolution and low-resolution grids agree.

Numerical Method: Projection

- The solution is advanced in time as follows:
- At the start of each time step j, the energies of the overlapping modes agree:

$$\frac{1}{2}\left|u_n^j\right|^2 + \frac{1}{2}|u_{n+1}^j|^2 = \frac{1}{2}|u_n^{(1)j}|^2\Delta_n.$$

- Using a Runge–Kutta integrator, the fine grid is advanced in time:

$$u_n^j \longrightarrow \widetilde{u}_n^{j+1} \quad u_{n+1}^j \longrightarrow \widetilde{u}_{n+1}^{j+1}.$$

- Next we *project* onto the coarse grid:

$$\widetilde{u}_n^{(1)j} = \sqrt{\frac{\left|\widetilde{u}_n^{j+1}\right|^2 + \left|\widetilde{u}_{n+1}^{j+1}\right|^2}{2}}.$$

Numerical Method: Prolongation

- Now we advance the coarse grid in time:

$$\widetilde{u}_n^{(1)j} \longrightarrow u_n^{(1)j+1}.$$

- Finally, we *prolong* from the coarse grid onto the fine grid:

$$u_n^{j+1} = \sqrt{\frac{|u_n^{(1)j+1}|^2}{|\widetilde{u}_n^{(1)j}|^2}}\,\widetilde{u}_n^{j+1}.$$

$$u_{n+1}^{j+1} = \sqrt{\frac{|u_n^{(1)j+1}|^2}{|\widetilde{u}_n^{(1)j}|^2}}\,\widetilde{u}_{n+1}^{j+1}.$$

- The projection and prolongation operators conserve energy whenever the two grids conserve energy in isolation.

- We can also include the changes in phase into the projection and prolongation operators, which may be important for Navier–Stokes turbulence.

Renormalisation of Shell Models

- In addition to energy conservation, the grids must relax at the same rate.
- We coarse-grain the equations by setting

$$u_n^{(1)} = \frac{u_{2n} + u_{2n+1}}{C}.$$

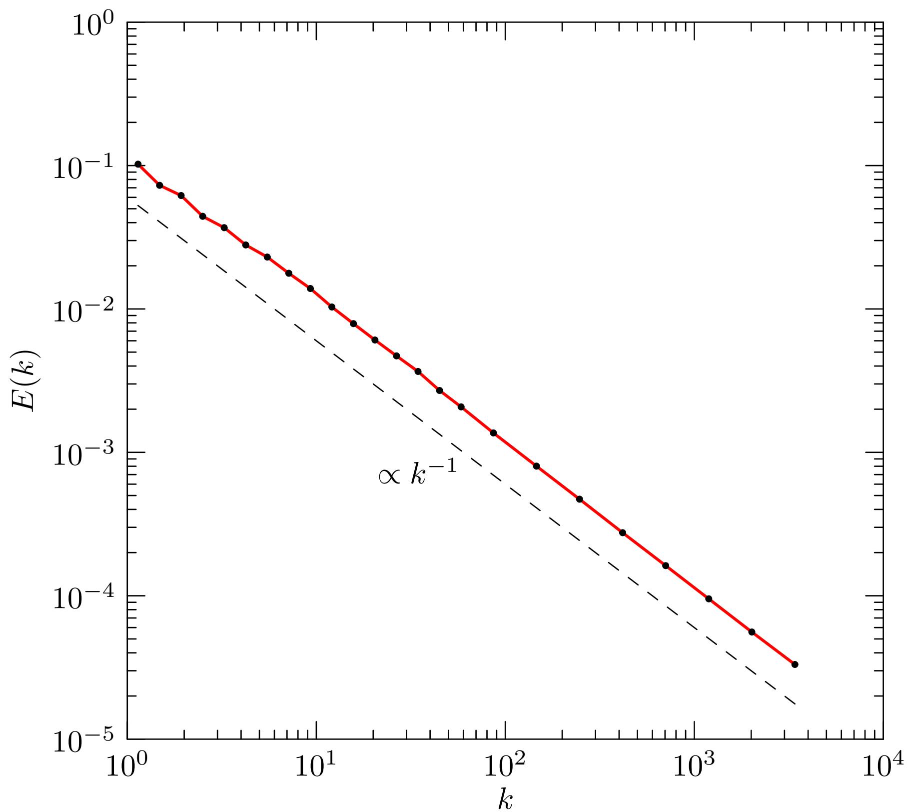
- The phases of u_{2n} and u_{2n+1} are uncorrelated, so

$$\left\langle \left| u_n^{(1)} \right|^2 \right\rangle = \frac{\left\langle \left| u_{2n} + u_{2n+1} \right|^2 \right\rangle}{C^2} = \frac{\left\langle \left| u_{2n} \right|^2 \right\rangle + \left\langle \left| u_{2n+1} \right|^2 \right\rangle}{C^2}$$

$$= \frac{\left\langle \left| u_{2n} \right|^2 \right\rangle + \left\langle \left| u_{2n+1} \right|^2 \right\rangle}{2} \Rightarrow C = \sqrt{2}$$

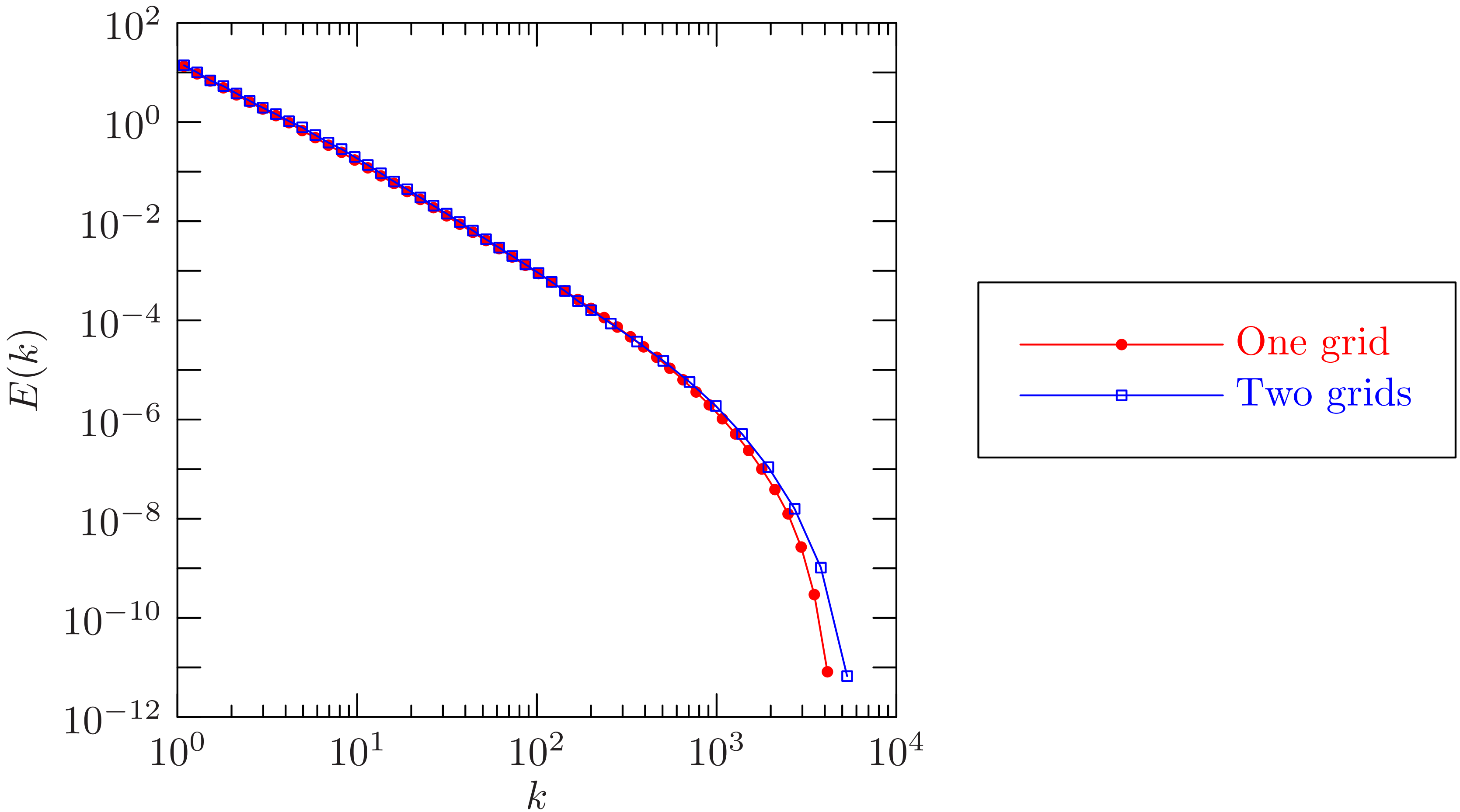
Multi-Spectral DN Model: Statistical-Mechanical Equipartition

- $\epsilon = \nu = 0$.
- In the absence of forcing and viscosity, all modes should have equal energy, giving a k^{-1} spectrum:



Multi-Spectral DN Model: Forced-Dissipative Turbulence

- $\epsilon = 1, \nu = 0.0001.$



- Full-resolution run is in red, decimated run in blue.

Conclusions

- Shell models are simple systems that can behave like Navier–Stokes turbulence.
- The multispectral method preserves behaviour of full-resolution simulation.
- Can be extended to a hierarchy of grids.
- Uses fewer modes than a full simulation (by a factor of $n2^{n-1}$).
- Future work: extension of reduction method to 2D and 3D Navier–Stokes dynamic subgrid model.

Asymptote: The Vector Graphics Language

`http://asymptote.sf.net`

(freely available under the GNU public license)

References

[Bell & Nelkin 1977] T. Bell & M. Nelkin, Phys. Fluids, **20**:345, 1977.

[Herweijer & van de Water 1995] J. Herweijer & W. van de Water, Phys. Rev. Lett., **74**:4651, 1995.

[Novikov 1964] E. A. Novikov, J. Exptl. Theoret. Phys. (U.S.S.R), **47**:1919, 1964.