Turbulence:
Analytic Results from Shell Models

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2008-05-10

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Outline

- The Renormalization Group
- Shell Models vs. Navier–Stokes
- High Resolution
  - With real velocities
  - With complex velocities
- Moments and Intermittency
- Conclusions

The Renormalization Group

- Explain RG theory

Shell Models vs. Navier–Stokes

- The Navier–Stokes equations:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \nu\nabla^2\boldsymbol{u} + \boldsymbol{F}.$$

- The NS equations set in Fourier space (2D vorticity formulation):

$$\left(\frac{\partial}{\partial t} + \nu_{\boldsymbol{k}}\right)\omega_{\boldsymbol{k}} = \int_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{0}} \frac{(\hat{\boldsymbol{z}}\boldsymbol{\cdot}\boldsymbol{p}\times\boldsymbol{q})}{p^2}\,\omega_{\boldsymbol{p}}^*\omega_{\boldsymbol{q}}^*\,d\boldsymbol{p} + F_{\boldsymbol{k}}.$$

- Shell models are one-dimensional caricatures of the spectral Navier–Stokes equations, having form

$$\left(\frac{\partial}{\partial t} + \nu k_n^2\right)u_n = \sum_{\ell,m} A_{\ell,m}u_\ell^*u_m^* + F_n.$$

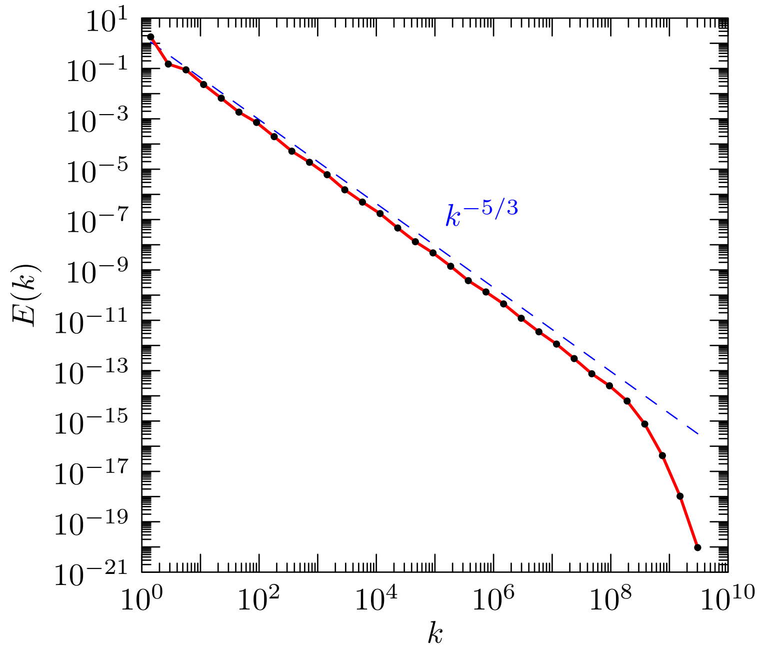
Shell Models Continued

- We look at the DN and GOY models:

$$\partial_t u_n + \nu\lambda^{2n} u_n = i\lambda^n \left[a\left(u_{n-1}^2 - \lambda u_n u_{n+1}\right) + b\left(\lambda u_{n+1}^2 - u_n u_{n-1}\right)\right]^*$$

$$\partial_t u_n + \nu\lambda^{2n} u_n = i\lambda^n \left(\alpha u_{n+1} u_{n+2} + \frac{\beta}{\lambda} u_{n-1} u_{n+1} + \frac{\gamma}{\lambda^2} u_{n-1} u_{n-2} \right)^*$$

- Wavenumbers: $k_n = \lambda^n$, energy: $E = \sum \frac{|u_n|^2}{k_{n+1} - k_n}$



High Resolution Limit of Shell Models

- Consider the DN and GOY models:

$$\partial_t u_n + \nu\lambda^{2n} u_n = i\lambda^n \left[a\left(u_{n-1}^2 - \lambda u_n u_{n+1}\right) + b\left(\lambda u_{n+1}^2 - u_n u_{n-1}\right)\right]^*$$

$$\partial_t u_n + \nu\lambda^{2n} u_n = i\lambda^n \left(\alpha u_{n+1} u_{n+2} + \frac{\beta}{\lambda} u_{n-1} u_{n+1} + \frac{\gamma}{\lambda^2} u_{n-1} u_{n-2}\right)^*$$

- Both systems use nearest or next-nearest neighbour interactions in terms of the shell index n.
- Set $\eta = n \log \lambda$.
- Use a series for neighboring modes, i.e.

$$u_{n+1} = u_n + \log \lambda \frac{\partial u_n}{\partial \eta}$$

High Resolution Limit of Shell Models, cont.

- Both the DN and GOY models achieve the same continuum limit (to first order in $\log\lambda$):

$$\left(\partial_t + \nu e^{2\eta}\right) u = i e^{\eta} K \log(\lambda) \left(u^2 + 3u\,\frac{\partial u}{\partial \eta}\right)^*,$$

where $K = b - a$ $(= 2\alpha + \beta)$ for the DN (GOY) model.

- Rescale the nonlinear coefficients by $1/\log\lambda$

$$\left(\partial_t + \nu e^{2\eta}\right) u = i e^{\eta} K \left(u^2 + 3u\,\frac{\partial u}{\partial \eta}\right)^*,$$

- so we achieve a fixed point in the mapping.

Case 1: Real Velocities

- Removing i and the complex conjugate, the continuous limit is

$$\left(\partial_t + \nu e^{2\eta}\right) u = e^{\eta} K \left(u^2 + 3u \frac{\partial u}{\partial \eta} \right),$$

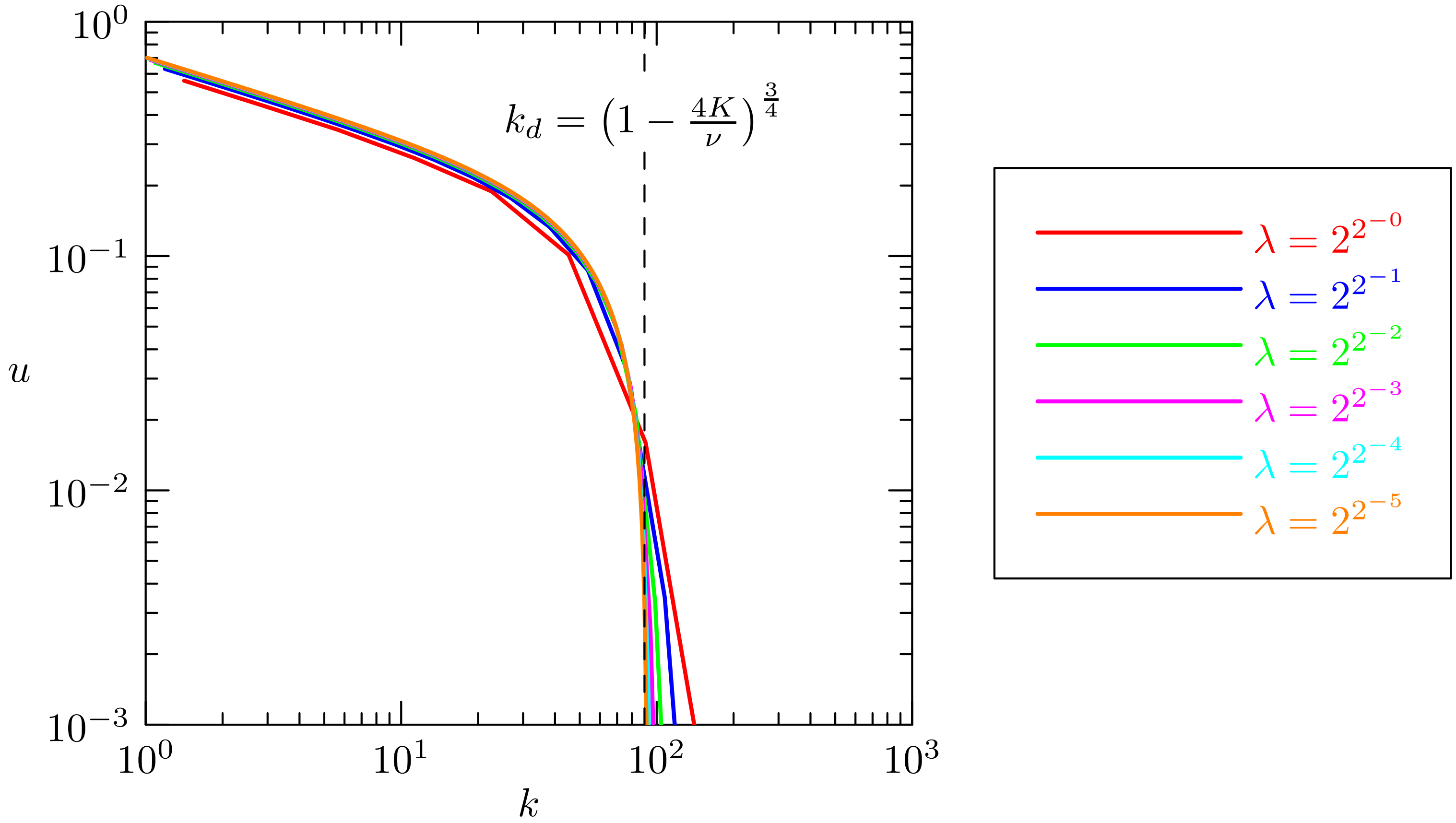
- which has steady-state solution

$$u(\eta) = \left(u_0 + \frac{\nu}{4K} \left(e^{\frac{4}{3}\eta} - 1 \right) \right) e^{-\frac{1}{3}\eta}.$$

- This reaches zero for finite η:

$$e^{\eta_d} = \left(1 - \frac{4 u_0 K}{\nu} \right)^{\frac{3}{4}}.$$

And we can reproduce it!



- Real velocities.
- $(a, b) = (-1, 0)$, constant boundary condition.

Structure Functions

- Define the structure functions S_p and exponents ζ_p

$$S_p = \left\langle u_k^p \right\rangle \propto k^{-\zeta_p}.$$

- Kolmogorov used some approximate arguments to get $\zeta_p = p/3$.

- We can calculate the moments exactly in this case:

$$\left\langle u_k^1 \right\rangle = e^{-\frac{1}{3}\eta} \left[c_1 + c_0 \frac{\nu}{3K} (1 - e^{\eta}) \right]$$

$$\left\langle u_k^2 \right\rangle = e^{-\frac{2}{3}\eta} \left[c_2 + c_1 \frac{\nu}{K} \left(1 - e^{-\frac{2}{3}\eta} \right) + c_0 \frac{\nu^2}{K^2} \left(\frac{1}{5} - \frac{1}{3} e^{\frac{2}{3}\eta} + \frac{2}{15} e^{\frac{5}{3}\eta} \right) \right]$$

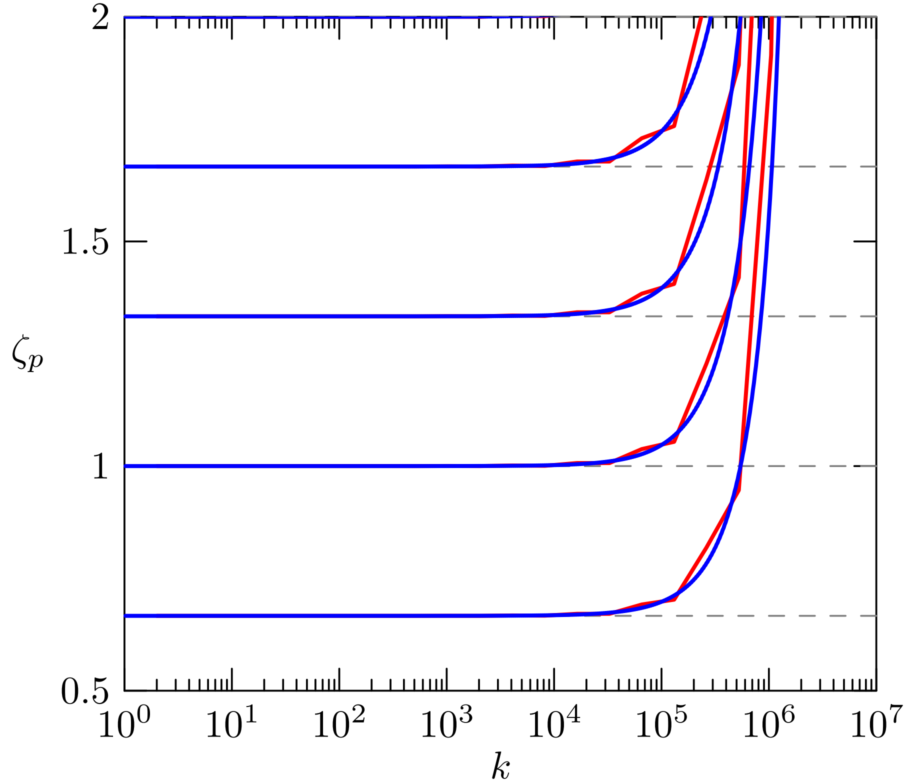
- Define the structure function exponents via

$$\zeta_p \doteq \left. \frac{dS^p/d\eta}{S^p} \right|_{\eta=0} = \frac{p}{3} \left(1 + \frac{\nu}{K} \frac{c_{p-1}}{c_p} \right)$$

- The dissipation has a non-zero value $\nu \longrightarrow 0$:

$$\lim_{\nu \to 0} 2\nu \int_0^{\kappa_d} k^2 E(k) dk = 3Kc_1c_2.$$

- Inviscid, unforced runs give $E(k) \propto k^{-2.5}$.
- Solutions are a fixed point, and do not reproduce intermittency.



Case 2: Complex Velocities

- The combination of i and complex conjugation in the non-linearity give energy conservation. The steady state is given by

$$\nu e^{\eta} u = iK \left(u^2 + 3u\partial_{\eta} u\right)^*$$

- which can be reduced to solving

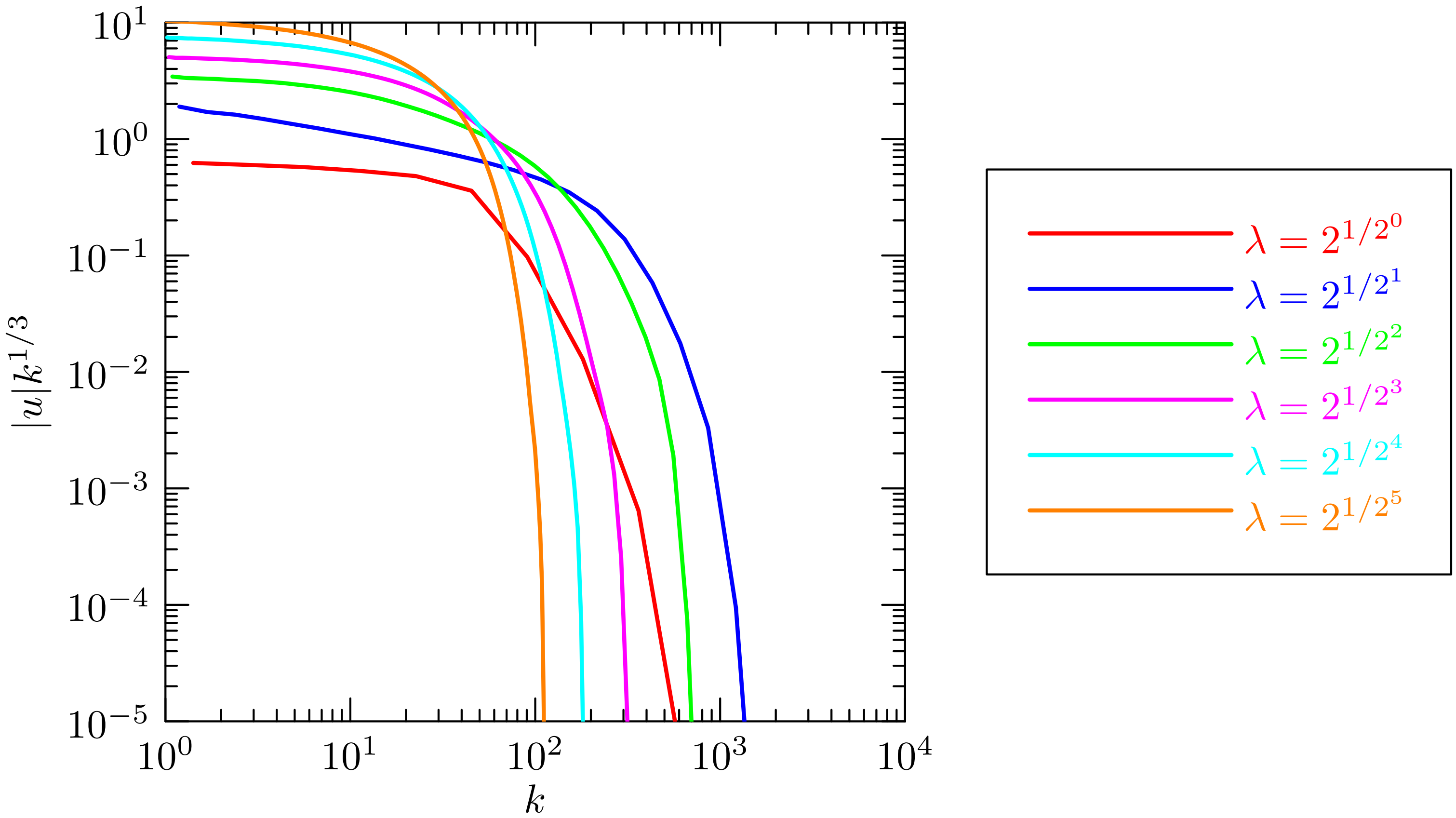
$$\frac{dv}{d\rho} = \frac{v^*}{v},$$

- the solution to which gives e^{η_d}, finite-ν corrections, and intermittency.

- unforced, inviscid simulations are in equipartition, with

$$E(k) \propto \frac{1}{k}.$$

- Further, the complex case doesn't rescale exactly like $\log \lambda$



- Complex velocities, i and c.c. in nonlinear term.
- $(a, b) = (-\frac{1}{4}, 1)$, constant boundary condition.

Conclusions and Further Work

- Confirmation of some of Kolmogorov's assumptions
- Predictions for finite-ν corrections.
- Future Work:
- Keep running simulations.
- Intermittency decreases with λ. Why?
- Look at turbulence onset
- Apply to Navier–Stokes turbulence.