Analytic Results from Shell Models of Turbulence

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Abstract

Analytic Results from Shell Models of Turbulence

Shell models of turbulence are simpler to deal with analytically and numerically than the full Navier–Stokes equations. In this work, we look the continuum limit of the DN and GOY shell models and reproduce results from Kolmogorov theory for the stationary case. The continuum limit allows us to derive these results analytically, which we also confirm numerically.

Outline

- Navier–Stokes Turbulence
- Shell Models of Turbulence
  - DN model
  - GOY model
- Continuum limit
  - Dissipation scale
  - Energy spectrum and moments
  - Inviscid dissipation limit

Navier–Stokes Turbulence

We are interested in answering questions about solutions to:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\boldsymbol{u} = -\left(\boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{u} - \frac{1}{\rho}\boldsymbol{\nabla}P + \boldsymbol{F} \qquad (1)$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$$

Kolmogorov theory gives us some predictions:

- The energy spectrum goes like $E(k) \propto k^{-\frac{5}{3}}$
- Let $\delta\boldsymbol{u}_{\|}(\boldsymbol{x},\boldsymbol{\ell}) = (\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{\ell}) - \boldsymbol{u}(\boldsymbol{x}))\cdot\hat{\boldsymbol{\ell}}$. Then $S_p(\ell) \doteq \left\langle\left|\boldsymbol{u}_{\|}(\boldsymbol{x},\boldsymbol{\ell})\right|\right\rangle \propto \ell^{\zeta_p}$ with $\zeta_p = p/3$.
- The dissipation scale: $\eta_d \propto \nu^{3/4}$
- The finite-viscosity limit.

Shell Models of Turbulence

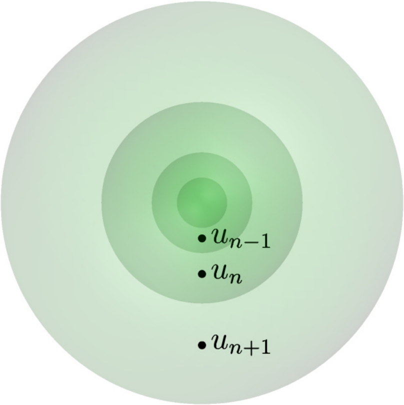
The Fourier transform of the N–S equation is

$$\frac{\partial \boldsymbol{u_k}}{\partial t} = \left(\mathbb{I} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}\right) \sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} i\left(\boldsymbol{k} \cdot \boldsymbol{u_p}\right)\boldsymbol{u_q} - \nu k^2 \boldsymbol{u_k} + \boldsymbol{F_k} \quad (2)$$

For shell models of turbulence, we represent all velocities $\{\boldsymbol{u_k}, k \in (k_{\text{min}}, k_{\text{max}})\}$ by a single complex velocity u_n.

$$\frac{\partial u_n}{\partial t} = k_n \sum_{\ell,m} A_{\ell,m} u_\ell^* u_m^* - \nu k_n^2 u_n + F_n \quad (3)$$

Shell Models of Turbulence



Generic Shell Model

Typically, $k_n = \lambda^n$ and u_n represents u_k for $k \in (k_n, k_{n+1})$.

We define analogous quantities

- $E \doteq \frac{1}{2} \sum |u_n|^2$
- $E(k_n) \doteq \frac{1}{2} \frac{|u_n^2|}{k_{n+1} - k_n}$
- $S_p \doteq \langle |u_n|^p \rangle = k_n^{\zeta_p}$

In general, the nonlinear term is restricted to nearby shells.

We consider shell models because

1. They share properties of the Navier–Stokes equations,
2. But they are simpler analytically and computationally.

DN Model

The DN model [Desnyansky & Novikov 1974] results from:

- nearest-neighbour interactions
- energy conservation

The evolution equation is

$$\frac{\partial u_n}{\partial t} = ik_n \left[a\left(u_{n-1}^2 - \lambda u_n u_{n+1}\right) + b\left(u_{n-1} u_n - \lambda u_{n+1}^2\right)\right] - \nu k_n^2 u_n + F_n \tag{4}$$

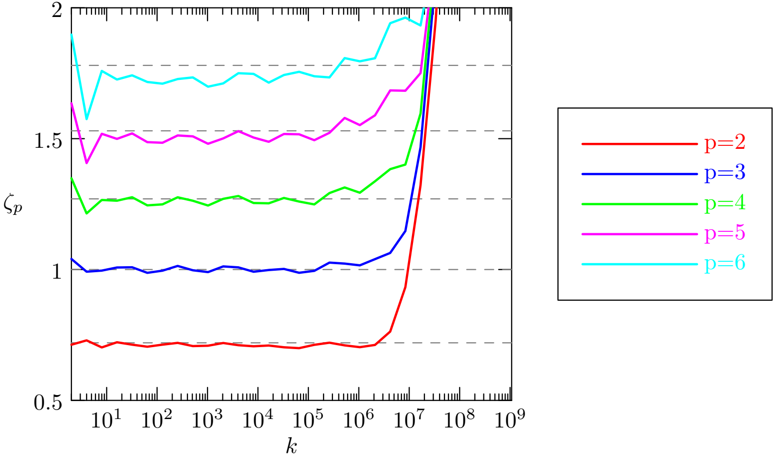
GOY Model

The GOY model ([Gledzer 1973], [Yamada & Ohkitani 1987]) extends this to next-nearest neighbour interactions.

$$\frac{\partial u_n}{\partial t} = ik_n \left(\alpha u_{n+1} u_{n+2} + \frac{\beta}{\lambda} u_{n-1} u_{n+1} + \frac{\gamma}{\lambda^2} u_{n-1} u_{n-2} \right)^* - \nu k_n^2 u_n + F_n. \tag{5}$$

- Energy is conserved if $\alpha + \beta + \gamma = 0$.
- A second helicity-like term $\frac{1}{2} \sum_n (-1)^n k_n |u_n|^2$ is conserved.

GOY Model: Structure Functions



Dashed lines are experimental values for water.

Continuum Shell Models

Shell models are discrete, and have few modes: they are easy to simulate.

We can fix this!

Let $\eta = n \log \lambda$

We use a Taylor series to for neighbouring modes, ie

$$u_{n+1} \approx u_n + \log \lambda \frac{\partial u_n}{\partial \eta} \tag{6}$$

We let $\lambda \to 0$ to get a continuum shell model.

Continuum Shell Models

Both the DN and GOY models have the same continuum limit:

$$\frac{\partial u}{\partial t} = -ie^{\eta}\hat{K}\log\lambda\left(u^2 + 3u\frac{\partial u}{\partial \eta}\right)^* - \nu e^{2\eta}u \tag{7}$$

where $\hat{K} = a - b$ for the DN model, and $\hat{K} = -2\alpha - \beta$ for the GOY model.

We rescale the nonlinear coefficient by $1/\log\lambda$:

$$\frac{\partial u}{\partial t} = ie^{\eta}\hat{K}\left(u^2 + 3u\frac{\partial u}{\partial \eta}\right)^* - \nu e^{2\eta}u \tag{8}$$

Continuum Shell Models: Real Case

If u is real-valued and positive, then we can drop the complex conjugate.

Let $K = -i\hat{K}$ be real-valued. Then

$$\frac{\partial u}{\partial t} = -e^{\eta} K \left(u^2 + 3u \frac{\partial u}{\partial \eta} \right) - \nu e^{2\eta} u \tag{9}$$

We can now solve the steady-state analytically:

$$u = \left[\frac{\nu}{4K} \left(1 - e^{\frac{4}{3}\eta} \right) + u_0 \right] e^{-\frac{\eta}{3}} \tag{10}$$

Dissipation Wavenumber

If u_0 is real-valued and positive, then

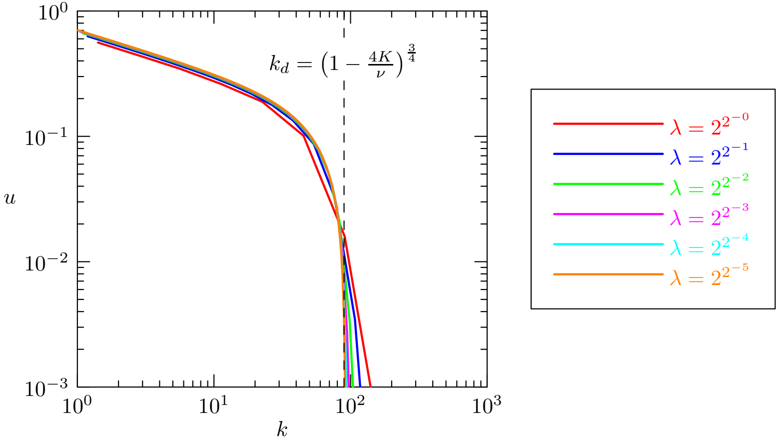
$$\frac{\nu}{4K}\left(1 - e^{\frac{4}{3}\eta}\right) + u_0 \tag{11}$$

is zero for some value of η, which we denote η_d.
If $\eta_d \gg 1$, then

$$k_d = e^{\eta_d} \approx \left(\frac{4Ku_0}{\nu}\right)^{\frac{3}{4}} \tag{12}$$

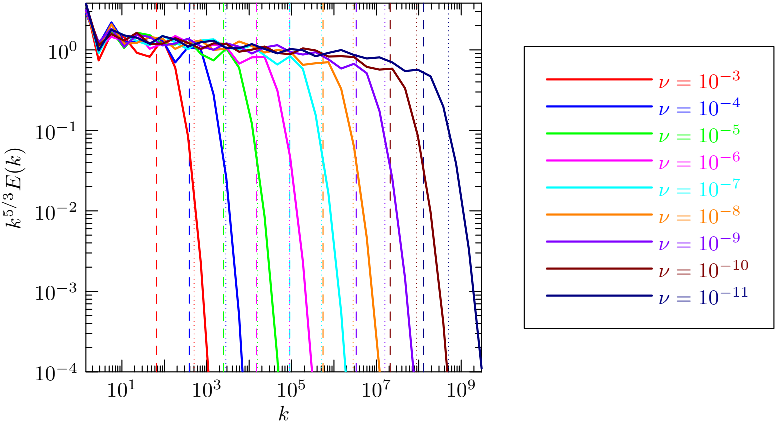
This reproduces Kolmogorov's prediction of $k_d \sim \nu^{-\frac{3}{4}}$.

Dissipation Wavenumber



Dissipation Wavenumber

Comparison of median dissipation wavenumber (dotted) and $(1 - 4K/\nu)^{3/4}$ (dashed).



Structure Functions

We multiply the continuum shell model by u^{p-2} and take a the time average $< \cdots >= \int dt$

$$\left\langle u^{p-2}\frac{\partial u}{\partial t}\right\rangle = Ke^{\eta}\left(\langle u^p\rangle + \frac{3}{p}\frac{\partial\,\langle u^p\rangle}{\partial\eta}\right) - \nu e^{2\eta}\left\langle u^{p-1}\right\rangle \quad (13)$$

We again take the steady state, setting $\frac{\partial u}{\partial t} = 0$.
Denote $c_p = \langle |u|^p\rangle\big|_{\eta=0}$.
Then,

$$\begin{aligned} S^1 &= \left\langle |u|^1 \right\rangle \\ &= e^{\eta/3}\left[c_0 + \frac{\nu}{3K}\left(e^{\frac{4}{3}}\eta - 1\right)\right] \end{aligned} \quad (14)$$

Structure Functions

Similarly,

$$\left\langle u^2 \right\rangle = e^{-\frac{2}{3}\eta}\left[c_2 + \frac{c_1}{2}\frac{\nu}{K}\left(e^{\frac{4}{3}\eta} - 1\right) + \frac{1}{6}\frac{\nu^2}{K^2}\left(e^{\frac{4}{3}\eta} - \frac{1}{2}e^{\frac{8}{3}\eta} - \frac{1}{2}\right)\right]$$

$$\begin{aligned}\left\langle u^3 \right\rangle = e^{\eta}\Bigg[c_3 - c_2\frac{3}{4}\frac{\nu}{K}\left(e^{\frac{4}{3}\eta} - 1\right) + c_1\frac{3}{8}\frac{\nu^2}{K^2}\left(e^{\frac{4}{3}\eta} - \frac{e^{\frac{8}{3}\eta}}{2} - \frac{1}{2}\right) \\ + \frac{1}{16}\frac{\nu^3}{K^3}\left(e^{\frac{4}{3}\eta} + e^{\frac{8}{3}\eta} - \frac{e^{4\eta}}{3} - \frac{5}{3}\right)\Bigg]\end{aligned}$$

$$\left\langle u^4 \right\rangle = e^{\frac{4}{3}\eta}\left[c_4 + \frac{\nu}{K}\frac{2}{5}c_3\left(1 - e^{\frac{10}{3}\eta}\right) + \ldots\right]$$

Structure Functions

Let

$$\zeta_p \doteq -\lim_{\eta\to 0} \frac{d\left\langle u^p\right\rangle/d\eta}{\left\langle u^p\right\rangle} \tag{15}$$

Then,

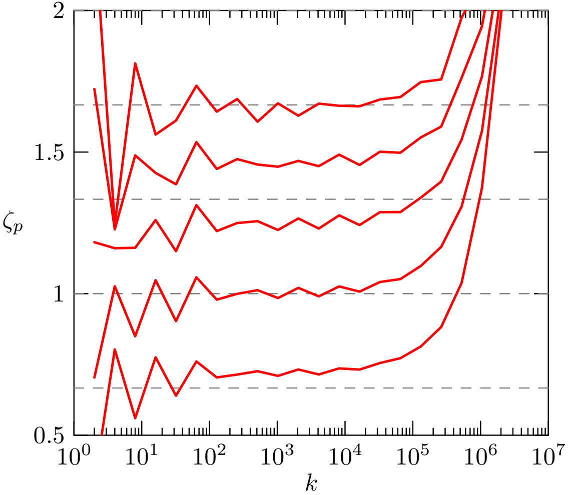
$$\begin{aligned}
\zeta_1 &= \frac{1}{3} + \frac{1}{c_1}\frac{4}{3}\frac{\nu}{K} && (16)\\
\zeta_2 &= \frac{2}{3} + \frac{c_1}{c_2}\frac{2}{3}\frac{\nu}{K} && (17)\\
\zeta_3 &= 1 - \frac{\nu}{K}\frac{c_2}{c_3} + \frac{1}{c_3}\frac{1}{6}\frac{\nu^2}{K^2} && (18)\\
\zeta_4 &= \frac{4}{3} - \frac{\nu}{K}\frac{c_3}{c_4}\frac{4}{3} + \frac{\nu^2}{K^2}\frac{c_2}{c_4}\frac{8}{315} + \mathcal{O}(\nu^3) && (19)
\end{aligned}$$

Structure Functions

Which gives us a finite-viscosity correction to the K41 values:

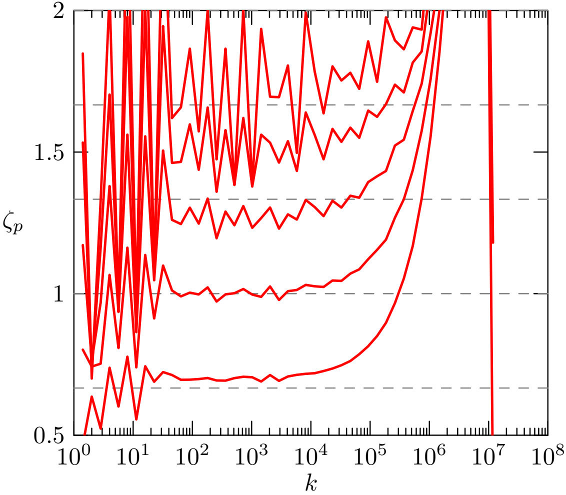
$$
\begin{aligned}
\zeta_1 &= \frac{1}{3} + \frac{1}{c_1}\frac{4}{3}\frac{\nu}{K} && (20)\\
\zeta_2 &= \frac{2}{3} + \frac{c_1}{c_2}\frac{2}{3}\frac{\nu}{K} && (21)\\
\zeta_3 &= 1 - \frac{\nu}{K}\frac{c_2}{c_3} && (22)\\
\zeta_4 &= \frac{4}{3} - \frac{\nu}{K}\frac{c_3}{c_4}\frac{4}{3} && (23)\\
&\cdots
\end{aligned}
$$

Structure Functions $\lambda = 2$



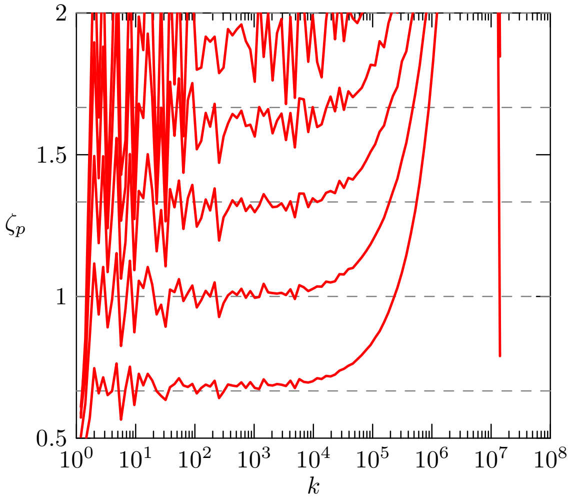
Dashed lines are K41 values.

Structure Functions $\lambda = 2^{1/2}$



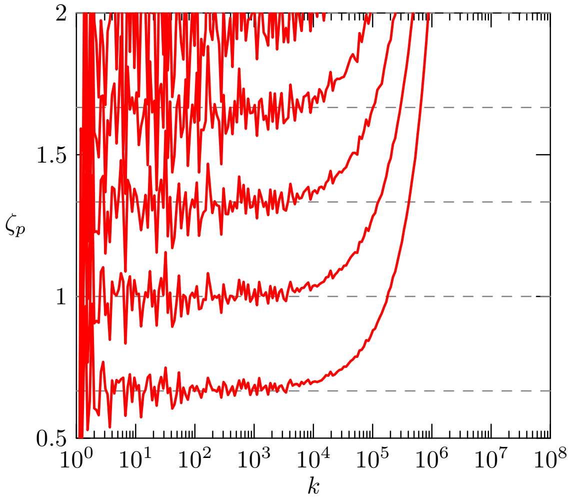
Dashed lines are K41 values.

Structure Functions $\lambda = 2^{1/2^2}$



Dashed lines are K41 values.

Structure Functions $\lambda = 2^{1/2^3}$



Dashed lines are K41 values.

Structure Functions and λ

What happened to anomalous scaling as $\lambda \to 1$?

For $\lambda = 2$

- Energy cascades from large to small scales via discrete jumps.

For $\lambda \to 1$

- Energy cascades from large to small scales smoothly.
- The role of u_{n+1} and u_{n+2} in the nonlinear term for u_n becomes more important: less back-scatter.

Inviscid Limit Dissipation

The energy dissipation rate is

$$\epsilon \doteq \left.\frac{dE}{dt}\right|_{\text{dissipative}} = -\,\nu \int e^{2\eta} u^2 \, d\eta. \tag{24}$$

We can calculate this for the steady state

$$\begin{aligned}\langle \epsilon \rangle &= -\nu \int_{\eta_0}^{\eta_d} e^{2\eta} \left\langle u^2 \right\rangle \, d\eta \\ &= -\nu \left[c_2 \frac{3}{4} e^{\frac{4}{3}\eta} + c_1 \frac{\nu}{K} \left(-\frac{3}{8} e^{\frac{4}{3}\eta} + \frac{3}{16} e^{\frac{8}{3}\eta} \right) \right. \\ &\qquad \left. + \frac{\nu^2}{K^2} \left(\frac{1}{48} e^{4\eta} - \frac{1}{16} e^{\frac{4}{3}\eta} + \frac{1}{16} e^{\frac{8}{3}\eta} \right) \right]_{\eta_0}^{\eta_d} . \end{aligned} \tag{25}$$

Inviscid Limit Dissipation

Let $\eta_0 = 0$, and use $e^{\eta_d} = \left(\frac{4Kc_0}{\nu}\right)^{4/3}$.

Then, take the limit as $\nu \to 0$.

We get

$$\lim_{\nu \to 0} \langle \epsilon \rangle = K \left(3c_2 c_1 + 80c_1^3\right) \neq 0 \tag{26}$$

reproducing Kolmogorov's finite-dissipation limit.

Conclusion

- The GOY and DN shell models have the same energy-preserving continuum limit.
- For the real-valued case we were able to derive:
  - The dissipation wavelength
  - Structure-function exponents
  - The zero-viscosity dissipation limit
- The results gave finite-viscosity corrections to Kolmogorov-style results.
- Anomalous scaling of the structure functions decreases as one approaches the continuum limit.

Thank you for your attention!

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